The localization game on Cartesian products

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Abstract

The localization game is played by two players: a Cop with a team of k cops, and a Robber. The game is initialised by the Robber choosing a vertex $r \in V$, unknown to the Cop. Thereafter, the game proceeds turn based. At the start of each turn, the Cop probes k vertices and in return receives a distance vector that indicates the distance from the Robber to each of the k vertices. If the Cop can determine the exact location of r from the vector, the Robber is located and the Cop wins. Otherwise, the Robber is allowed to either stay at r, or move to r' in the neighbourhood of r. The Cop then again probes k vertices. The game continues in this fashion, where the Cop wins if the Robber can be located in a finite number of turns. The localization number $\zeta(G)$, is defined as the least positive integer k for which the Cop has a winning strategy irrespective of the moves of the Robber.

In this thesis, the focus falls on the localization game played on Cartesian products. Upper and lower bounds on the localization number of two arbitrary graphs are established, where the concept of doubly resolving sets are used for the upper bound. When the Cartesian product of an arbitrary graph with a complete graph is considered, the localization number is at most the largest of the orders of the graphs. This bound is achieved when both graphs are complete graphs. The exact values of the localization number of the Cartesian product of complete graphs with cycles and paths are also established.

The exact values of the localization number of the Cartesian product of two cycles as well as a cycle with a path are determined and an upper bound on the localization number of the Cartesian product of an arbitrary graph and a cycle is presented.

Lastly the Cartesian products of stars are investigated. The exact value of the localization number of the product of two stars is established, showing that the difference between the localization number of G and the localization number of the Cartesian product of two copies of G can be arbitrarily large. It is also illustrated that if the localization number of G is less than that of H, it does not imply that the localization number of $G \square G$ is less than that of $H \square H$.

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Uittreksel

In grafiekteorie word die opsporingspeletjie deur twee spelers gespeel: 'n Polisieman met 'n span van k polisiemanne, en 'n Skurk. Die speletjie begin deur die Skurk wat 'n node $r \in V$ kies, onbekend aan die Polisieman. Hierna gaan die speletjie beurtsgewys voort. Aan die begin van elke beurt kies die Polisieman k nodusse en ontvang daarna 'n afstandsvektor wat die afstand vanaf die Skurk na elk van die k nodusse aandui. As die Polisieman van die afstandsvektor kan aflei presies waar die Skurk is, dan is die Skurk opgespoor en die Polisieman wen. Andersins word die Skurk toegelaat om óf te bly by r, óf te skuif na r' in die omgewing van r. Hierna kan die Polisieman weer k nodusse kies. Die speletjie gaan op hierdie manier voort, waar die Polisieman wen as die Skurk in 'n eindige aantal beurte opgespoor kan word. Die opsporingsgetal $\zeta(G)$ is die kleinste heelgetal k waarvoor die Polisieman definitief kan wen, ongeag van die Skurk se strategie.

In hierdie tesis val die fokus op die opsporingspeletjie wat op die Cartesiese produk van grafieke gespeel word. Bo- en ondergrense van die opsporingsgetal van twee arbitrêre grafieke word bepaal, waar die konsep van dubbeloplossingsversamelings gebruik word vir die bogrens. Wanneer die Cartesiese produk van 'n arbitrêre grafiek met 'n volledige grafiek beskou word, is die opsporingsgetal op die meeste die grootste van die twee ordes. Hierdie grens word behaal wanneer beide grafieke volledig is. Die eksakte waarde van die opsporingsgetal van die Cartesiese produk van volledige grafieke met siklusse en paaie word ook gevind.

Die eksakte waarde van die opsporingsgetal van die Cartesiese produk van twee siklusse, asook van 'n siklus en 'n pad, word bepaal en 'n bogrens op die opsporingsgetal van die Cartesiese produk van 'n arbitrêre grafiek met 'n siklus word gegee.

Laastens word die Cartesiese produk van sterre ondersoek. Die eksakte waarde van die opsporingsgetal van die produk van twee sterre word gevind en sodoende word daar bewys dat die verskil tussen die opsporingsgetal van G en die opsporingsgetal van die Cartesiese produk van twee kopieë van G arbitrêr groot kan wees. Daar word ook gewys dat as die opsporingsgetal van G kleiner is as die van H, dit nie impliseer dat die opsporingsgetal van $G \square G$ kleiner is as die van $H \square H$ nie.

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CHAPTER 1

Introduction and basic definitions

1.1 Introduction

The localization game is a variant of the game of Cops and Robbers and was independently introduced in 2018 by Bosek et al. [5] and by Haslegrave et al. [15]. Bosek et al. was inspired by localization problems in wireless networks. Consider a mobile phone connected to a Wi-Fi network. The closer the phone is to the Wi-Fi router, the stronger the Wi-Fi signal received by the phone, but without the knowledge in which direction the router is placed. Can the phone user determine where exactly the router is placed if they only have the distance to the router? What if the router is moved while this attempted localization is underway? And what if multiple phone users work together to locate the router?

The game is played on a simple, connected, undirected graph $G = (V, E)$. Two players are involved in this game: a Cop who has a team of k cops, and a Robber. To start the game, the Robber chooses a vertex $r \in V$, unknown to the Cop. After this, the game proceeds turn based. At the start of each turn, the Cop probes k vertices $B = \{b_1, b_2, \ldots, b_k\}$. In return, the Cop receives the vector $\vec{D}(\{r\}, B) = [d_1, d_2, \ldots, d_k]$ where d_i is the distance in G from r to b_i for $i = 1, 2, \ldots, k$. If the Cop can determine the exact location of r from $\vec{D}(\{r\}, B)$, the Robber is located and the Cop wins. Otherwise, the Robber is allowed to either stay at r, or to pick a new vertex r' adjacent to vertex r. The Cop then again probes k vertices. These k vertices are allowed to be the same as in previous turns. The game continues in this fashion, where the Cop wins if the Robber can be located in a finite number of turns. If the Cop fails to locate the Robber in a finite number of turns, the Robber wins. If the Cop correctly guesses the location of the Robber, then d_i is zero for some $i = 1, 2, \ldots, k$ and the Robber is located. The localization number $\zeta(G)$, is defined as the least positive integer k for which the Cop has a winning strategy irrespective of the moves of the Robber. Thus the Cop will locate the Robber in a finite number of turns, even if the Robber knows the Cop's strategy beforehand.

1.2 Basic definitions

A *graph* $G = (V, E)$ is nonempty, finite set $V(G)$ of elements called *vertices*, together with a possibly empty set $E(G)$ of pairs of vertices, called *edges*. The *order* of G is the number of vertices in the graph G and the *size* is the number of edges of graph G. If it clear from the context, $V(G)$ and $E(G)$ are denoted by V and E respectively. A graph G of order m will be denoted by G_m. The edge between vertices $v_1, v_2 \in V$ is denoted by v_1v_2, where v_1 and v_2 are called *adjacent* and v_1 or v_2 is *incident* to edge v_1v_2. A vertex that is adjacent to every other

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vertex, is called an *universal vertex*.

The *open neighbourhood* $N(v)$ of vertex $v \in V(G)$ is the set of all vertices adjacent to v. The *closed neighbourhood* $N[v]$ is equal to $N(v) \cup \{v\}$. The *degree* of a vertex is the cardinality of its open neighbourhood. The *minimum degree* of a graph G is denoted by $\delta(G)$ and is defined as the smallest degree among all the vertices of G. Similarly, the *maximum degree* is denoted by $\Delta(G)$ and is defined as the largest degree among all vertices of G. A graph where each vertex has the same degree p, is called a p-regular graph. The *neighbourhood* of the set $S \subseteq V(G)$ is defined as the union of all $N(s)$ for $s \in S$, denoted by $N(S)$ and $N[S]$ is the union of all $N[s]$ for $s \in S$.

A *walk* of length k is an alternating sequence $W = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges where $e_i = v_{i-1}v_i$ for $i = 0, 1, \ldots, n$. If $v_0 = x$ and $v_n = y$, then W is called an $x - y$ walk of length n. A *path* is a walk where all the vertices are distinct. A graph of order n that consists only of a path is called the *path of order* n and is denoted by P_n. If a walk W has $v_0 = v_n$ and all other vertices are distinct, the walk is called a *cycle* of length n. If a graph of order n consists only of a cycle, it is called the *cycle of order* n and denoted by C_n. The cycle C_n is called an *even cycle* if n is even and an *odd cycle* otherwise. The *girth* of a graph is the length of a shortest cycle contained in the graph.

The *distance* $d(v_i, v_j)$ between two vertices v_i and v_j is the length of a shortest path between them. This path is then called a $v_i - v_j$ *geodesic*. A graph is *connected* if there exists a path from any vertex to any other vertex and *disconnected* otherwise. If a graph is disconnected, then its vertex set can be divided into *components*, where a component is a maximal connected subgraph of G.

A *graph property* or *graph invariant* is a property of a graph that is only dependent on the abstract structure of the graph and not on representations like vertex labeling or drawing. A graph property P is *hereditary* if every induced subgraph of a graph with property P also has the property P. Further a graph property P is *monotone* if every subgraph of a graph with property P also has the property P. Note that if a property is monotone, then it is also hereditary.

A *complete graph* of order n is denoted by K_n and is defined such that every possible edge exists or equivalently such that each vertex has degree $n-1$. A *bipartite graph* G is a graph where $V(G)$ can be partitioned into partite sets U and W such that $V = U \cup W$, where $uw \in E(G)$ only if $u \in U$ and $w \in W$. If every possible edge in a bipartite graph exists, then it is called a *complete bipartite graph* and denoted by $K_{a,b}$ where $|U| = a$ and $|W| = b$. The complete bipartite graph $K_{1,m}$ is also called a *star* and is also denoted by S_{m+1}. The *Cartesian product* $G \square H$ of two graphs G and H is a graph with vertex set the Cartesian product $V(G) \times V(H)$. Further two vertices (u, u') and (v, v') in $G \square H$ are adjacent if and only if either $u = v$ and $d_H(u', v') = 1$, or $u' = v'$ and $d_G(u, v) = 1$. Note that in this thesis, the "Cartesian product" will sometimes be referred to as simply the "product".

A set of vertices $S \subseteq G$ is a *resolving set* of graph G if every vertex in G is uniquely defined by its distance to the vertices in S. The *metric dimension* $\dim(G)$ of a graph G is defined as the minimum cardinality of a set $S \subseteq G$ such that S resolves G.

Further, an *automorphism* of a graph $G = (V, E)$ is a permutation σ of the vertex set V such that uv is an edge of G if and only if $\sigma(u)\sigma(v)$ is an edge of G. For a vertex v of G, the set of all vertices into which v can be mapped by some automorphism of G is an *orbit* of G. Two vertices in the same orbit are called *similar*.

The *chromatic number* $\chi(G)$ is defined as the least number of colours needed to colour each vertex in $V(G)$ such that if two vertices are adjacent, they are different colours.

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1.3 Thesis layout

This thesis contains seven chapters (including this chapter).

In Chapter 2, a literature review is given on the localization game and other related games. A short overview of the game of Cops and Robbers is given first. Thereafter, results on the robber locating game and the backtrack robber locating game are discussed. Known results on the localization game are reviewed in Section 2.4, with a focus on exact values of Cartesian graph classes and general bounds. The chapter closes with an overview of the centroidal localization game.

The localization game in general is considered in Chapter 3. At first, an example game and basic results are given. Then the localization number of special graph classes, that is complete graphs, cycles and grids, are considered. The localization number of general Cartesian products is investigated in Section 3.3. A novel lower and upper bound is provided for the localization number of Cartesian products. Results on the doubly resolving number are also provided.

In Chapter 4 products with complete graphs are considered. An upper bound to the localization number of the product of a complete graph with any graph is established in Section 4.1. Furthermore, the localization number of the product of two complete graphs is determined. Lastly, the product of a complete graph with a cycle is investigated.

In Chapter 5, the Cartesian product of cycles is considered. Specifically, the localization number of the product of two cycles is found by considering three cases: odd by odd, odd by even and even by even. The chapter ends with the investigation of the product of a general graph with a cycle. The localization number of the product of a path and a cycle is determined and an upper bound to the localization number of the product of any graph with a cycle is provided.

The product of two star graphs is considered in Chapter 6. The focus falls on calculating the localization number of the product of two star of the same order.

In the last chapter a summary of work done in this thesis is given as well as some ideas with respect to future work on the localization number Cartesian graph products.

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CHAPTER 2

Literature Review

A literature review on the localization game and related pursuit-evasion games is provided in this chapter. The chapter begins with an introduction to the game of Cops and Robbers after which the Robber locating game, both with and without backtracking is discussed. The chapter concludes with results on the localization game and the centroidal localization game.

2.1 Cops and Robbers

The game of Cops and Robbers was studied as early as 1983 by Nowakowski et al. in [20]. The game involves two players: a Cop and a Robber and is played on an undirected, connected graph G. The game starts with the Cop occupying some vertex of G. The Robber then also chooses a vertex to occupy, after which the Cop attempts to catch the Robber. The two players take turns moving, where a move consists of moving to a neighbouring vertex of the previously occupied vertex. The Cop wins if the Robber is caught in a finite number of turns. This happens when at some point the Cop occupies the same vertex as the Robber. The *cop number* $c(G)$ of the graph is defined as the least amount of moves needed for the Cop to guarantee a win. Since the game has perfect information, one of the players will always win. Graphs can therefore also be divided into *cop-win* graphs and *robber-win* graphs.

Note that different to the localization game, the game of Cops and Robbers, is played with perfect information. *Perfect information* means that each player has all the information of events that previously occurred [21]. An example of a game with perfect information is Chess, because at each turn both players know what the other player's moves were before the turn. In the case of Cops and Robbers, this means that both players can see all the moves of the other player. An example of a game with imperfect information is Texas hold'em poker, since players cannot see each other's cards. The localization game has imperfect information, since the Cop cannot see where the Robber is. Note that there exists variations of the game of Cops and Robbers that *are* played with imperfect information, as in [11], [16] and [18].

As in literature, assume the Cop to be male and the Robber female.

2.2 The robber locating game

In 2012, Seager [23] combined the game of Cops and Robbers and the concept of metric dimension by introducing the robber locating game.

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The robber locating game starts with the Robber choosing some vertex $r_1 \in V(G)$ unknown to the Cop. The Cop then probes a vertex b_1 and receives the distance $d(b_1, r)$ in return. If this distance uniquely defines the location of the Robber, the Cop wins. If not, the Robber is allowed to either stay at r_1, or move to a vertex that neighbours it and is not equal to the previously probed vertex b_1. The Cop then again probes some vertex b_2 and receives $d(b_2, r_2)$ in return. If this does not uniquely define the location of r_2, the Robber can move to any vertex $r_3 \in N[r_2]\backslash\{b_2\}$. The game continues in this fashion, where the Cop wins if the Robber is located in a finite number of turns. At the end of every turn, the Robber is allowed to move to any neighbouring vertex, excluding the one previously probed by the Cop. A graph is *locatable* if the Cop can guarantee a win in a finite number of turns. The *location number*, denoted by $\mathrm{loc}(G)$, is the least number of turns needed to do this. The aim of this game is therefore to determine if a graph G is locatable and if so, what its location number is. Note that the localization game is more closely related to whether a graph is locatable than its location number.

Seager showed that a graph is locatable with $\mathrm{loc}(G) = 1$ if and only if G is a path. She also showed that K_3 and $K_{2,3}$ are locatable, where any graph with K_4 as a subgraph is not. Further if a graph has $K_{3,3}$ as an induced subgraph, then it is not locatable. The cycle C_n is locatable for $n = 4$ and $n > 5$, but not for $n = 5$. She also showed that all trees are locatable and calculated the location number for different types of trees.

In 2014, Johnson et al. [17] showed that the graph property of being locatable is not closed under edge or vertex removal. This proved that no forbidden subgraph or induced subgraph characterisation of locatable graphs exist. However, a characterization of non-locatable diameter two graphs was provided. They showed that every locatable graph is four-colourable and described subgraphs where the Robber can hide from the Cop.

2.3 The backtrack robber locating game

Carraher et al. [9] removed the restriction on the Robber's movement that disallowed moving to the previously probed vertex. They called this restriction the *no-backtrack* condition. Note that this version of the game is harder for the Cop and therefore if a graph is not locatable in the robber locating game, it is also not locatable in the backtrack robber locating game. Further if the Cop can win in the backtrack robber locating game, the Cop can win in the robber locating game. They showed that the Robber wins on any graph containing a cycle of length at most five.

Then in 2014, Seager [24] investigated this version of the game as well. She also showed that the Cop wins on all cycles of order $n > 6$. Let $T_{3,3}$ be the tree on ten vertices where one vertex has three neighbours and each of these neighbours is adjacent to two leaves. Seager showed that the Cop wins on a tree if and only if it does not contain a copy of $T_{3,3}$. Brandt et al. [7] further investigated the location number of trees in 2017, providing a strategy to locate the Robber on a tree. This strategy many times needed less turns than the one provided by the bound in [24].

2.4 The localization game

The localization game was independently introduced in 2018 by Bosek et al. [5] and by Haslegrave et al. [15]. In these papers the backtracking robber locating game was extended to allow the Cop to probe a set of k vertices every turn such that a distance vector is received in stead of a single distance. Haslegrave et al. showed that for any integer k, if the Cop can win using k cops,

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then $V(G)$ is countable. They also provided the following bounds on the localization number in terms of the maximum degree:

Proposition 2.1 [15]. *For a connected graph G with maximum degree Δ, the following inequality holds:*

$$\zeta(G) \leq \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor + 1.$$

Proposition 2.2 [15]. *There exists a connected graph G with maximum degree Δ such that $\zeta(G) \geq \left\lfloor \frac{\Delta^2}{4} \right\rfloor$.*

Proposition 2.3 [15]. *For any connected graph G with $\Delta(G) = 3$, $\zeta(G) \leq 3$.*

A *path-decomposition* of a graph G is a sequence $X = (X_1, X_2, \ldots, X_t)$ of subsets of $V(G)$, called bags, such that for every edge $uv \in E(G)$ the following holds:

- There exists a bag containing both u and v and
- for every $1 \leq i \leq k \leq j \leq t$ it is true that $X_i \cap X_j \subseteq X_k$.

The *width* of the sequence X is equal to $\max_{1 \leq i \leq t} |X_i| - 1$ and the *pathwidth* of G is the minimum width of its path decompositions. The following bound in terms of pathwidth was proved by Bosek et al. [5]:

Proposition 2.4 [5]. *For connected graph G with pathwidth $pw(G)$, $\zeta(G) \leq pw(G)$. This bound is achieved for interval graphs.*

In the above result, an *interval graph* is an undirected graph from the real intervals S_i for $i = 0, 1, 2, \ldots$. This is done by creating a vertex v_i for each interval S_i and connecting two vertices v_i and v_j whenever the corresponding two sets have a nonempty intersection such that $E(G) = \{v_i v_j \mid S_i \cap S_j \neq \emptyset\}$. Bosek et al. further showed that for paths, complete graphs and stars the following holds:

$$\zeta(P_n) = \dim(P_n) = 1, \tag{2.1}$$
$$\zeta(K_n) = \dim(K_n) = n - 1 \text{ and} \tag{2.2}$$
$$\zeta(S_n) = 1,\ \dim(S_n) = n - 1. \tag{2.3}$$

They also provided the following results regarding bipartite graphs:

Proposition 2.5 [5]. *For complete bipartite graph $K_{a,b}$, $\zeta(K_{a,b}) = \min\{a, b\}$.*

Corollary 2.6 [5]. *Let G be a bipartite graph with partite set sizes a and b respectively. Then $\zeta(G) \leq \min\{a, b\}$.*

An example that illustrated that the localization number is not monotone on taking subgraphs was also presented in [5].

A variation of the localization game where the cops are *blind* was introduced in [5]. The game proceeds as the localization game, with the difference that the Cop does not receive a distance vector after the probe. Instead, the Cop merely knows whether the Robber was at a probed vertex or a neighbour of a probed vertex. The Cop wins if this is the case. The smallest number of cops needed to win is denoted by $\zeta_b(G)$.

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Proposition 2.7 [5]. *For a given graph G, let G' be a copy of G with one additional vertex v adjacent to all vertices of G. Then $\zeta_b(G) \leq \zeta(G')$.*

By using this variation they showed that $\zeta(G)$ is unbounded for planar graphs:

Proposition 2.8 [5]. *For any $k > 0$, there exists a planar graph G with treewidth 2 (precisely, a tree plus an universal vertex) such that $\zeta(G) > k$.*

In the above result, a *planar graph* is defined as a graph that can be drawn on the plane such that no two edges cross each other. Further, a graph G is *outerplanar* if the graph formed from G by adding a universal vertex is a planar graph. Even though $\zeta(G)$ is unbounded for planar graphs, it is bounded for outerplanar graphs:

Proposition 2.9 [3]. *If G is an outerplanar graph, then $\zeta(G) \leq 2$.*

Let the degeneracy of a graph G be defined as the maximum, over all subgraphs H of G, of $\delta(H)$. Bosek et al. provide the following bounds for graphs with degeneracy k:

Proposition 2.10 [3]. *If G is a graph of degeneracy k, then $\zeta(G) \geq \log_3(k+1)$.*

Corollary 2.11 [3]. *For every graph G with chromatic number $\chi(G)$, we have that $\chi(G) \leq 3^{\zeta(G)}$.*

Corollary 2.12 [3]. *If G is a bipartite graph of degeneracy k, then $\zeta(G) \geq \log_2 k$.*

Bosek et al. considered Cartesian products of paths and the hypercube where $Q_n = K_2 \Box Q_{n-1}$ and $Q_0 = K_1$:

Proposition 2.13 [3]. *For hypercube Q_n and all positive integers n, the following holds: $\zeta(Q_n) \leq \lceil \log_2(n-1) \rceil + 3$.*

Proposition 2.14 [3]. *If $G = G_0 \Box G_1 \Box \ldots \Box G_{n-1}$, where each G_i is a path, then $\zeta(G) \leq \lceil \log_2 n \rceil + 2$.*

In the latter result, note that the Cartesian product of graphs is associative.

The localization number of dense random graphs were studied in [13] and [14], while Bonato et al. considered diameter two graphs [1] as well as the game played on designs [2]. Determining the localization number of an arbitrary graph has been determined to be NP hard by Bosek et al. in [5].

One can naturally extend the game to the Euclidean plane. For this, the infinite graph G_1 was defined whose vertices are all points on the plane with edges between points at Euclidean distance at most one. Bosek et al. then proved the following:

Proposition 2.15 [5]. *Let $\aleph_0$ denote the cardinality of the natural numbers. Then $\zeta(G_1) > \aleph_0$.*

In view of this, they relaxed the game such that the Cop receives the Euclidean distance to the Robber, calling it the geometric localization game. Then the following holds true:

1. Three cops can win in one round.
2. Two cops can win in two rounds.
3. One cop cannot win in any number of rounds.

Proposition 2.16 [5]. *For $\epsilon > 0$, one cop can locate the Robber with error at most $1 + \epsilon$. In other words, one cop can determine a disk of radius $1 + \epsilon$ in which the Robber is contained.*

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2.5 Centroidal localization game

A variation of the localization game, called the centroidal localization game, was also introduced Bosek et al. [4]. It proceeds the same as the localization game, with the difference that the Cop does not receive a distance vector. Instead, for a probe $\{v_1, v_2, \ldots, v_k\}$, he receives for any $1 \leq i < j \leq k$ one of the following:

- whether $d(v_i, r) = 0$, or
- $d(v_i, r) = d(v_j, r) \neq 0$, or
- $d(v_i, r) < d(v_j, r)$, or
- $d(v_i, r) > d(v_j, r)$.

The Cop wins if this information uniquely defines the location of the Robber. Note that the Cop can win without probing the exact vertex of the Robber. The *centroidal localization number* $\zeta^*(G)$ is the smallest number of Cops needed to guarantee a win, such that $\zeta(G) \leq \zeta^*(G)$. The results on the centroidal localization number of the Cartesian product of graphs can be extended to the localization game:

Proposition 2.17 [4]. *For any two graphs G and H, the following holds:*

$$\zeta(G \square H) \leq \max\{\Delta(G) + \Delta(H) + 1, \Delta(G) + \zeta(H), \zeta(G) + \Delta(H)\}.$$

2.6 Chapter summary

This chapter started with an introduction to the game of Cops and Robbers, a predecessor to the localization game. In Section 2.2 the robber locating game was introduced and locatable and non-locatable graphs were discussed. Graph where a single cop wins the localization game were investigated in the backtrack robber locating game. The localization game is reviewed in Section 2.4. Bounds on the localization number are presented as well as the localization number of specific graph classes. The chapter concluded with results on the centroidal localization game.

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CHAPTER 3

The localization game

Upper and lower bounds on the localization number of the Cartesian product of graphs are established in this chapter. This chapter starts off with some basic results on the localization number of special graph classes, specifically complete graphs, cycles and grids. In Section 3.3 a general lower and upper bound on $\zeta(G\square H)$ is provided. This chapter concludes with a discussion on the doubly resolving number of a graph, which provides an upper bound on the localization number.

3.1 Example game and basic results

For a warm-up exercise, let's determine the localization number of $K_{2,3}$.

Example 3.1 The localization number of $K_{2,3}$. Let $G = K_{2,3}$ with partite sets given by $V = \{v_1, v_2, v_3\}$ and $U = \{u_1, u_2\}$. We show that $\zeta(G) = 2$.

Proof. First say the Cop plays with one cop and probes $B_1 = \{b_1\}$ in the first turn. If the Robber chooses to be at a vertex r in a different partite set to b_1, then the Cop will receive distance vector $\vec{D}(B_1, r) = [1]$. Without loss of generality, say the Cop probe $b_1 \in U$ such that $r \in V$. This will localize the Robber to any vertex in V such that the Cop has not located the Robber. In the next turn, the Robber can either stay at r, or move to any vertex in U. If the Cop probes $B_2 = \{b_2\}$ such that $b_2 \in U$, then the Robber stays at r. If the Cop instead probes $b_2 \in V$, the Robber moves to a vertex in U. In both cases the Cop receives the distance vector [1] as in the first turn. The Robber can therefore perpetually avoid detection by insuring that she is located in the partite set not probed by the Cop and thus it follows that $\zeta(G) > 1$.

Next, say the Cop plays with two cops and probes $B_1 = \{v_1, u_1\}$ in the first turn as illustrated on the left in Figure 3.1. In the figure, square vertices are probed, red vertices are safe for the Robber, lighter red vertices are neighbours of the safe vertices and empty vertices are resolved by the probe. From the distances in the figure it is clear that the Robber is only safe at two vertices: v_2 and v_3. Assume the Robber is at either of these vertices, i.e., $\vec{D}(B_1, r) = [2, 1]$. The Robber can now either stay at r, or move to a neighbour such that she can be at any vertex in G in the second turn except vertex v_1. This is illustrated on the middle figure in Figure 3.1.

In the second turn, the Cop probes $B_2 = \{v_2, u_1\}$ such that the distances to the safe vertices are given on the right of Figure 3.1. Since every vertex where the Robber can be is resolved, the Cop wins and $\zeta(G) \leq 2$. □

The following result proves that the localization number is well defined, since $\zeta(G) \leq |V(G)|$. The proof is given as an introduction to the game.

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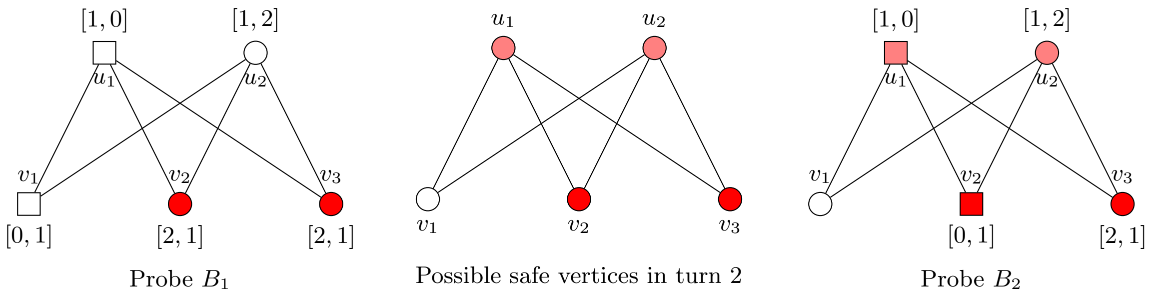

FIGURE 3.1: *The localization game on $K_{2,3}$. Square vertices are probed, red vertices are safe for the Robber, lighter red vertices are neighbours of the safe vertices and empty vertices are resolved by the probe.*

Proposition 3.1 [5]. *Let G be a graph of order n. Then $\zeta(G) \leq n-1$.*

Proof. Let $k = n-1$. Say the Cop probes $B = V(G) \setminus \{v\}$ for some $v \in V$. If $r \in B$, then $d_i = 0$ for some $i = 1, 2, \ldots, k$ and the Robber has been located. If not, then $r = v$ and the Robber has also been located. This means that the Cop will win after one turn and

$$\zeta(G) \leq k = n-1$$

since $\zeta(G)$ is the smallest k for which the Cop has a winning strategy. □

The following result is mentioned in [5] without proof and shows that only connected graphs need to be considered:

Proposition 3.2 [5]. *Let H be a disconnected graph. Then*

$$\zeta(H) = \max_i\{\zeta(H_i)\}$$

where H_i is a component of H.

Proof. Let $x = \max_i\{\zeta(H_i)\}$ and call the component of H in which this occurs M. Note that this means that for all H_i, $x \geq \zeta(H_i)$ and $x = \zeta(M)$.

Say the Cop plays with $k < x$ cops. In the first turn, assume the Robber chooses r to be in M. If the Cop probes vertices in other components than M, the distances will merely tell the Cop that r is not in the probed component. If the Cop probes any amount of vertices in M, the Robber will not be located since $k < x = \zeta(M)$. Thus the Robber will never be located if r is chosen to be in M and

$$\zeta(H) \geq x.$$

Let $k = x$ and r be in any component of H. Consider the strategy where the Cop probes vertices such that all the vertices probed in a turn are in the same component of H. The distances that the Cop will receive after a probe will then tell the Cop whether r is located in the component which was probed. If r is not in that component, the Cop probes another component in the next turn until the component containing r has been found. Note that the Robber cannot move between components, since by definition there does not exist a path from r to another component of H. If the Cop only probes vertices in the component in which r is located, the Robber will be located since for all H_i, $k = x \geq \zeta(C)$. Thus

$$\zeta(H) \leq k = x$$

and hence

$$\zeta(H) = x = \max_i\{\zeta(H_i)\}.$$ □

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The metric dimension of G can equivalently be defined as the smallest positive integer k such that the Cop locates the Robber in one turn and hence

$$\zeta(G) \leq \dim(G). \tag{3.1}$$

The localization number of G can be less than the metric dimension, since a smaller set than the set S with minimum cardinality could still possibly locate the Robber in more than one turn. As shown by Equations (2.1) and (2.2), $\zeta(G) = \dim(G)$ for paths and complete graphs. However as shown by Equation (2.3), if G is a star, the difference between $\dim(G)$ and $\zeta(G)$ can be arbitrarily large.

Bosek et al. [5] mention that in general, $\zeta(G)$ is not monotone on taking subgraphs. They give the example of $F = K_4$ and H being formed from F by adding two vertices and four edges. Let $V(F) = \{v_1, v_2, v_3, v_4\}$ and form H by adding vertices u, w with u being adjacent to v_1, v_2 and w being adjacent to v_2, v_3, see Figure 3.2. It is easy to check that $\zeta(F) = 3$, but $\zeta(H) = 2$ by probing u and w and thus proving that the localization number is not monotone on taking subgraphs.

Now remove the edge wv_3 from H to form G. Note that $V(G) = V(H)$, but $\zeta(G) = 3$ and $\zeta(H) = 2$. Thus in general, $\zeta(G)$ is not monotone on removing edges from a graph.

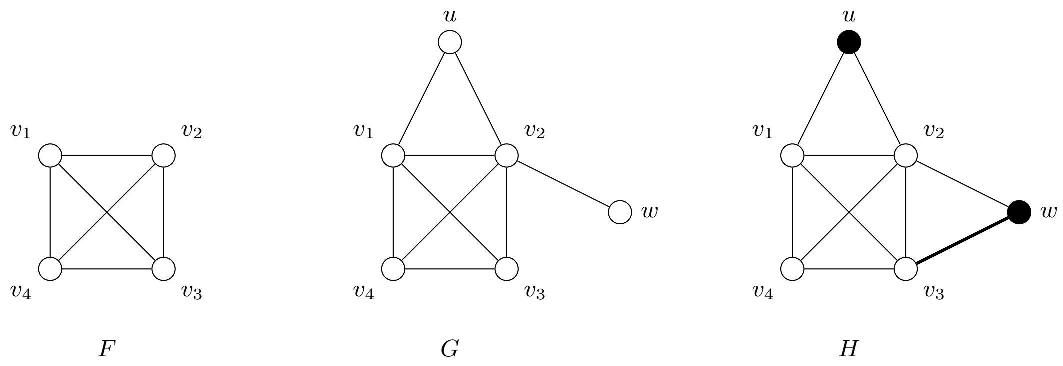

FIGURE 3.2: *Three graphs such that $F \subset G \subset H$ and $\zeta(F) = \zeta(G) = 3$, where $\zeta(H) = 2$.*

Lemma 3.3 [19]. *Let $G = (V, E)$ be an arbitrary graph. Let u, v, w be vertices of G and let $uv \in E$. Let d be the length of a shortest path from u to w in G. Then the length of a shortest path from v to w is one of $\{d-1, d, d+1\}$.*

Proposition 3.4. *If a graph G has minimum degree $\delta(G) \geq 3$, then $\zeta(G) > 1$.*

Proof. Say the Cop probes the set $B = \{b\}$ in some turn and let $d = d(b, v)$ where $v \in V(G)$. Note that $|N[v]| \geq 4$ since $\deg(v) \geq 3$. By Lemma 3.3, each vertex in $N[v]$ can be one of three distances: $d-1$, d or $d+1$. Therefore there are only three distances possible for at least four vertices and by the pigeonhole principle, at least two of these vertices will be the same distance from B. The Robber can therefore perpetually avoid capture by applying the following strategy:

- Before the Cop's first probe, the Robber chooses to be restricted to the neighbourhood $N[v]$ for some vertex v.
- By the above result, there will be at least two vertices u and w the same distance from the Cop's probe. The Robber then chooses to occupy any of these two vertices, say vertex u.

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- In subsequent probes, the Robber repeats this strategy by choosing to be restricted to movement in $N[u]$. □

The technique in the above proof is based on the technique used by Bonato et al. [3] in their proof of Proposition 2.10.

Corollary 3.5. *Let G be any connected 3-regular graph. Then $\zeta(G) \in \{2,3\}$.*

Proof. By Proposition 3.4, it is known that $\zeta(G) \geq 2$. Proposition 2.3 showed that if $\Delta(G) = 3$, $\zeta(G) \leq 3$. Together this proves that the localization number of 3-regular graphs can either be two or three. □

Using Propositions 2.1 and 2.3, we can also derive the following corollary:

Corollary 3.6. *Let G be a connected graph with maximum degree $\Delta(G)$. Then $\zeta(G) \geq 4$ if and only if $\Delta(G) \geq 4$.*

3.2 Localization number of special graph classes

3.2.1 Complete graphs

Lemma 3.7 [10]. *A connected graph G of order $n \geq 2$ has dimension $n-1$ if and only if $G = K_n$.*

The localization number of complete graphs is given by Bosek et al. [5] without proof.

Proposition 3.8 [5]. *Let K_n be the complete graph with n vertices. Then $\zeta(K_n) = \dim(K_n) = n-1$.*

Proof. By Proposition 3.1 it only needs to be shown that $\zeta(K_n) \geq n-1$. Consider the localization game where the Cop plays with $n-2$ cops. Both unprobed vertices are at distance one away from all probed vertices and hence the Robber cannot be located in the first turn. In the next turn the Robber can move to any vertex in K_n, putting the Cop in the same position as in the first turn. The Robber can therefore perpetually avoid capture if $n-2$ cops are used such that $\zeta(K_n) \geq n-1$. □

3.2.2 Cycles

From Khuller et al. [19] it is known that $\dim(C_n) = 2$ where C_n is a cycle of order n. Together with Equation (3.1), this proves that $\zeta(C_n) \leq 2$. By Proposition 3.8, it is known that $\zeta(C_3) = \dim(C_3) = 2$. Seager [23] proved that the Cop can win using only one cop for $n \geq 7$, if the Robber is not allowed to move to the previous vertex probed by the Cop. In [24], Seager noted that her proof in [23] can easily be adapted to prove the following lemma:

Lemma 3.9 [24]. *Let C_n be a cycle or order $n \geq 7$. Then $\zeta(C_n) = 1$.*

Definition 3.1 Hideout [24]. A *hideout* is defined as a subgraph H of G where the robber can win by remaining on the vertices of H.

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Lemma 3.10 [9]. *Let G be any graph containing a cycle of length at most five, where the localization game is played with one cop. Then this cycle is a hideout such that $\zeta(G) \neq 1$.*

However, not all graphs of girth six have localization number greater than one:

Proposition 3.11 [24]. *Let G be a graph of girth six and let C be a cycle of length six in G, such that no edge of C is contained in an odd cycle of G. Then $\zeta(G) \neq 1$.*

Corollary 3.12. *Let G be the cycle of length six. Then $\zeta(G) \neq 1$.*

Lemmas 3.9 and 3.10 together with Corollary 3.12 prove the following general result about the localization number of cycles:

Theorem 3.13. *Let C_n be the cycle of order n. Then $\zeta(C_n) = 2$ for $n \leq 6$ and $\zeta(C_n) = 1$ for $n \geq 7$.*

In conclusion note that even cycles are bipartite graphs and since cycles are 2-regular, C_n has degeneracy $k = 2$. By Corollary 2.12, $\zeta(C_n) \geq \log_2 2 = 1$ and by Theorem 3.13, $\zeta(C_n) = 1$. Therefore even cycles of order at least eight prove the tightness of Corollary 2.12 and so providing an affirmative answer to the question in [3] on whether the bound is tight.

3.2.3 Grids

Let $G_{m,n} = P_m \square P_n$ be a grid of order mn. Vertices will be labeled $v_{i,j}$ for $i \in \{0, 1, \ldots, m-1\}$ and $j \in \{0, 1, \ldots, n-1\}$ such that $v_{0,0}$ is the bottom left vertex and the grid is embedded on the positive quadrant of a Cartesian coordinate system. Further, for vertex $v_{i,j} \in G \square H$, we say that $v_{i,j}$ corresponds to vertex $g_i \in G$ and $h_j \in H$. As an example $G_{4,3}$ is given in Figure 3.3.

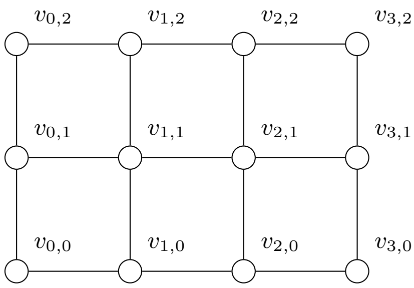

FIGURE 3.3: *The grid $G_{4,3}$.*

Note that in the case where m or n are equal to one, a path is obtained. If $n = 1$, then $G_{m,1} = P_m$ and $\dim(P_m) = \zeta(P_m) = 1$ [5].

Lemma 3.14 [19]. *For $d \geq 2$, the metric dimension of a d-dimensional grid is d.*

Theorem 3.15. *Let $G_{m,n}$ be a grid with $m, n \geq 2$. Then $\dim(G_{m,n}) = \zeta(G_{m,n}) = 2$.*

Proof. By Lemma 3.14, the dimension of $G_{m,n}$ is two and therefore $\zeta(G_{m,n}) \leq 2$. Note that the grid $G_{m,n}$ contains a cycle of length four and thus by Lemma 3.10, $\zeta(G_{m,n}) \geq 2$. $\square$

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3.3 General Cartesian products

Recall that the Cartesian product $G\Box H$ of two graphs G and H is a graph with the Cartesian product $V(G)\times V(H)$ as vertex set. Further two vertices (u,u') and (v,v') in $G\Box H$ are adjacent if and only if either $u=v$ and $d_H(u',v')=1$, or $u'=v'$ and $d_G(u,v)=1$. A *column* of $G\Box H$ is a set of vertices $\{(v,v'):v'\in V(H)\}$ for some vertex $v\in V(G)$ and a *row* of $G\Box H$ is a set of vertices $\{(v,v'):v\in V(G)\}$ for some vertex $v'\in V(H)$.

Definition 3.2 Safe vertex. A vertex v is called a safe vertex if it is not uniquely defined by probe B. In other words, there exists another vertex w that is the same distance from B as v.

Definition 3.3 Safe set. A safe set is a set of safe vertices that are all the same distance from B. By definition, every safe vertex is part of a safe set.

Definition 3.4 Robber set [24]. The robber set is defined as the safe set that the Robber has been localized to and is denoted by O_α in turn α. In the next turn, the Robber can move to any vertex in $N[O_\alpha]$.

Lemma 3.16 [22]. *Consider the graphs X and Y. Then $\chi(X\Box Y)=\max\{\chi(X),\chi(Y)\}$.*

Corollary 2.11 states that $\chi(G)\le 3^{\zeta(G)}$, which can equivalently be written as $\zeta(G)\ge\log_3(\chi(G))$. Lemma 3.16 further states that $\chi(X\Box Y)=\max\{\chi(X),\chi(Y)\}$, providing the following lower bound for the localization number of graph products:

Proposition 3.17. *Let G and H be any graphs. Then $\zeta(G\Box H)\ge\log_3\left(\max\{\chi(G),\chi(H)\}\right)$.*

Note that even though the chromatic number of a graph provides a lower bound for the localization number, these two quantities are not generally proportional. This is illustrated by cyclic grids $C_{2p}\Box C_4$ and $C_{2p+1}\Box C_{2q+1}$: by Lemma 3.16, $\chi\left(C_{2p}\Box C_4\right)=2<\chi\left(C_{2p+1}\Box C_{2q+1}\right)$, but by Theorem 5.1 $\zeta(C_{2p+1}\Box C_{2q+1})=2<3=\zeta(C_{2p}\Box C_4)$.

Proposition 3.18 [22]. *The product of connected graphs is connected. The product of any graph by a disconnected graph is disconnected.*

As shown in Proposition 3.2, only connected graphs need to be considered and therefore it may be assumed that $G\Box H$ is connected. Note that if G and H are connected, then $d\left((u,u'),(v,v')\right)=d_G(u,v)+d_H(u',v')$. The following theorem provides a lower bound for $\zeta(G\Box H)$ and is tight by Theorem 3.15.

Proposition 3.19. *Let G and H be any connected graphs of orders at least two. Then $\zeta(G\Box H)\ge 2$.*

Proof. Since G and H have orders at least two, the graph $G\Box H$ is not a path. Consider any vertex (u,u') in $G\Box H$. Since graphs G and H are connected, $\deg_G(u)\ge 1$ and $\deg_H(u')\ge 1$ and therefore (u,u') is adjacent to at least two vertices, say vertices (u,v') and (w,u') where $d_G(u,w)=1$ and $d_H(u',v')=1$. It follows that (w,v') is adjacent to (u,v') and (w,u'). Therefore a Cartesian product $G\Box H$ always contains a 4-cycle if G and H have orders at least two and are connected. By Lemma 3.10, $\zeta(G\Box H)>1$. $\Box$

The *imagination strategy* was introduced by Brešar et al. [8] in 2010 for the domination game on graphs. The idea of the imagination strategy is that one of the players imagines another

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appropriate game and plays in it according to a known winning strategy. As an example, say the localization game is played on some graph G. Assume the Cop plays by using the imagination strategy, where a graph G' is imagined such that a winning strategy is known for the Cop on graph G'. The Cop therefore has a probe B_1' on graph G' which will lead to the Cop locating the Robber in a finite number of turns. This probe is copied to G such that the Cop probes B_1 in the first turn. The Cop next receives some distance vector $\vec{D}(B_1, r)$ and copies this to graph G'. Again a second probe B_2' on G' is known, which is copied to the graph G such that B_2 is probed. The game continues in this fashion. It is possible that a probe by the Cop in the imagined game is not legal in the real game and it is also possible that the distance received by the Cop in the real game does not exist in the imagined game. Both these problems need to be considered when using this strategy.

Definition 3.5 Projections [12]. Let S be a set of vertices in the Cartesian product $G\Box H$. The *projection* of S onto G is the set of vertices $v \in V(G)$ for which there exists a vertex $(v, v') \in S$. Similarly, the *projection* of S onto H is the set of vertices $v' \in V(H)$ for which there exists a vertex $(v, v') \in S$.

Theorem 3.20. *For any two graphs G and H, the following equation holds:*

$$\zeta(G\Box H) \geq \max\{\zeta(G), \zeta(H)\}.$$

Proof. Consider the localization game played on the Cartesian product $G\Box H$. Say the Cop plays with $k = \zeta(G) - 1$ cops and that the Robber plays by imagining the localization game on G. In the first turn, the Robber occupies some vertex r_1 in the imagined game. In the real game, the Robber chooses to occupy vertex (r_1, j) for some row j in $G\Box H$. In the turns to follow, the Robber applies the following strategy: Say in turn α the Cop probes $B_\alpha = \{b_1, b_2, \ldots, b_k\}$. Let S_α be the projection of B_α onto G, such that S_α contains at most k vertices. The Robber then imagines the Cop probes S_α on graph G, where the Robber is always able to avoid capture since $|S_\alpha| \leq k < \zeta(G)$. If the Robber moves to vertex $r_{\alpha+1}$ in the imagined game, he moves to vertex $(r_{\alpha+1}, j)$ in the real game. The games continues in this fashion such that the Cop never wins and $\zeta(G\Box H) > k = \zeta(G) - 1$. In a similar fashion it can be shown that $\zeta(G\Box H) > \zeta(H) - 1$ and thus $\zeta(G\Box H) \geq \max\{\zeta(G), \zeta(H)\}$. □

Definition 3.6 Doubly resolving sets [12]. Let $G \neq K_1$ be a graph. Two vertices $v_1, v_2 \in V(G)$ are *doubly resolved* by vertices $u_1, u_2 \in V(G)$ if

$$d(v_1, u_1) - d(v_2, u_1) \neq d(v_1, u_2) - d(v_2, u_2).$$

A set $W \subseteq V(G)$ *doubly resolves* G and is a *doubly resolving set*, if every pair of distinct vertices $v_1, v_2 \in V(G)$ are doubly resolved by two vertices in W. A doubly resolving set with the smallest cardinality is denoted by $\psi(G)$.

Even though $\psi(G)$ is defined in [12], it is never named and hence we name it the *doubly resolving number* of a graph G. Every graph G with at least two vertices has a doubly resolving set and therefore it is well defined. Note that when calculating if some set $W \subseteq V(G)$ is a doubly resolving set, the vertex pairs inside W need not be considered. To prove this, consider any two distinct vertices $w_1, w_2 \in W$. Clearly $d(w_1, w_1) - d(w_2, w_1) = -d(w_2, w_1)$ where $d(w_1, w_2) - d(w_2, w_2) = d(w_1, w_2)$ so w_1, w_2 are doubly resolved by W. Cáceres et al. proved that $2 \leq \psi(G) \leq m - 1$ for any graph G of order $m \geq 3$, where it was also shown that $\dim(G) \leq \psi(G)$. They also proved the following proposition:

Proposition 3.21 [12]. *For all graphs G and $H \neq K_1$,* $\dim(G\Box H) \leq \dim(G) + \psi(H) - 1$.

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Lemma 3.22 [12]. *Let $S \subseteq V(G\Box H)$ for graphs G and H. Then every pair of vertices in a fixed column of $G\Box H$ is resolved by S if and only if the projection of S onto H resolves H. Similarly, every pair of vertices in a fixed row of $G\Box H$ is resolved by S if and only if the projection of S onto G resolves G.*

Corollary 3.23. *Let B be a probe on $G\Box H$ such that a safe set exists. This safe set will contain two vertices in the same column if and only if the projection of B onto H is not a resolving set of H. Furthermore, the projection of this safe set onto H will be the safe set in H after probing B's projection.*

Similarly, a safe set in $G\Box H$ will contain two vertices in the same row if and only if the projection of B onto G is not a resolving set of G. The projection of this safe set will again be equal to the safe set in G after probing B's projection on G.

The following theorem is analogous to Proposition 3.21:

Theorem 3.24. *Let G and H be any connected graphs, where $\zeta(G)$ and $\psi(H)$ is known. Then $\zeta(G\Box H) \leq \zeta(G) + \psi(H) - 1$.*

Proof. It needs to be shown that the Cop can win on $G\Box H$ using κ cops, where $\kappa = \zeta(G) + \psi(H) - 1$. To this end, the Cop imagines the localization game on graph G. Let T be a doubly resolving set of H such that $\psi(H) = |T|$. Further, say the Cop probes B_1 in the first turn of the imagination game such that $|B_1| = \zeta(G)$. For a fixed $b_1 \in B_1$ and $t \in T$, define a set X_1 such that $X_1 \coloneqq \{(b_1, t^i) : t^i \in T\} \cup \{(b_1^i, t) : b_1^i \in B_1\}$. Note that $|X_1| = \kappa$ and each entry of X_1 is a vertex in $G\Box H$. In the first turn in the real game, the Cop probes X_1. It will now be shown that any safe set for this probe is contained in a single row of $G\Box H$ and further that the projection of this safe set onto G is a valid safe set in G. Consider two distinct vertices (g, h) and (g', h') of $G\Box H$ where

$$\vec{D}\left((g,h), X_1\right) = \vec{D}\left((g',h'), X_1\right). \tag{3.2}$$

Since T is a doubly resolving set, the projection of X_1 onto H resolves H by Lemma 3.22. Therefore Equation (3.2) does not hold if $g = g'$. If $h \neq h'$, then there exists two vertices $t_k, t_l \in T$ such that

$$d_H(h, t_k) - d_H(h', t_k) \neq d_H(h, t_l) - d_H(h', t_l) \tag{3.3}$$

since T is a doubly resolving set of H. Equation (3.2) implies that

$$d_{G\Box H}\left((g,h),(x,x')\right) = d_{G\Box H}\left((g',h'),(x,x')\right)$$

for any $(x, x') \in X_1$. Thus

$$\begin{aligned} d_{G\Box H}\left((g,h),(b_1,t_k)\right) &= d_{G\Box H}\left((g',h'),(b_1,t_k)\right) \text{ and} \\ d_{G\Box H}\left((g,h),(b_1,t_l)\right) &= d_{G\Box H}\left((g',h'),(b_1,t_l)\right) \end{aligned}$$

such that

$$d_G(g, b_1) + d_H(h, t_k) = d_G(g', b_1) + d_H(h', t_k) \text{ and} \tag{3.4}$$

$$d_G(g, b_1) + d_H(h, t_l) = d_G(g', b_1) + d_H(h', t_l). \tag{3.5}$$

Equations (3.4) and (3.5) together imply

$$d_H(h, t_k) - d_H(h', t_k) = d_H(h, t_l) - d_H(h', t_l),$$

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contradicting Equation (3.3) and therefore Equation (3.2) only holds if $h = h'$. It follows that $d_G(g, b_1) = d_G(g', b_1)$ for any $b_1 \in B_1$ such that vertices (g, h) and (g', h') are in the same safe set in $G\square H$ if and only if vertices g and g' are in the same safe set in the imagination game.

Say the Robber is localized to robber set O_1 in $G\square H$, where Q_1 is the projection of O_1 onto G. It has been shown that O_1 is contained in a single row and that Q_1 is a valid robber set in the imagination game. For robber set Q_1 in the imagination game, a probe B_2 is known such that the Cop wins in a finite number of turns. For a fixed $b_2 \in B_2$, let $X_2 \coloneqq \{(b_2, t^i) : t^i \in T\} \cup \{(b_2^i, t) : b_2^i \in B_2\}$ such that $|X_2| = \kappa$. It can again be shown that two vertices (a, b) and (a', b') in $N[O_1]$ only belong to the same safe set in the real game if $b = b'$ and if a and a' belong to the same safe set in the imagination game. However since only vertices inside $N[O_1]$ are considered, it is not true that all sets $\{(a, b), (a', b)\}$ are safe sets in $G\square H$ if the set $\{a, a'\}$ is a safe set in G. Say the robber is localized to O_2 in the real game and localized to Q_2 in the imagination game. Then O_2 will be contained in a single row and its projection onto G will either be equal to Q_2, or a subset of Q_2. Therefore the Cop can imagine the robber set Q_2 on G such that B_3 is probed. The Cop continues in this fashion until the Robber is located. This is guaranteed because in some turn s on graph G, the robber set Q_s will only contain one vertex and therefore the robber set O_s in the real game will also only contain one vertex. Note that in turn τ, the projection of $N[O_\tau]$ onto G will be contained in $N_G[Q_\tau]$ and therefore the imagination strategy is valid in every turn. □

Corollary 3.25. *Let G and H be any connected graphs. By restricting $\zeta(G)$ or $\psi(H)$, we get the following results:*

1. *If $\zeta(G) = 1$, then $\zeta(H) \leq \zeta(G\square H) \leq \psi(H)$.*
2. *If $\psi(H) = 2$, then $\zeta(G) \leq \zeta(G\square H) \leq \zeta(G) + 1$.*
3. *If $\zeta(G) = 1$ and $\psi(H) = 2$, then $\zeta(G\square H) = 2$.*

Note that Theorem 3.24 is a significant improvement on Proposition 2.17 by Bosek et al. [4] in some cases. As an example, let $G = H = S_m$ where $\zeta(S_m) = 1$ and $\Delta(S_m) = m - 1$. Then Proposition 2.17 implies that $\zeta(S_m\square S_m) \leq 2m - 1$, where Theorem 3.24 implies $\zeta(S_m\square S_m) \leq m - 1$.

Let G_m be any connected graph of order m and P_n a path of order n, Since $\zeta(P_n) = 1$, it follows from Corollary 3.25 that

$$\zeta(G_m\square P_n) \leq \psi(G_m) \leq m - 1. \tag{3.6}$$

Cáceres et al. [12] showed that $\dim(G_m\square P_n) \leq \dim(G_m) + 1 \leq m$ such that $\zeta(G_m\square P_n) \leq m$. However it follows that $\dim(G_m\square P_n) \leq m - 1$ if G_m is not a complete graph, since $\dim(G_m) = m - 1$ if and only if $G_m = K_m$ by Lemma 3.7. Thus the bound of Equation (3.6) is not an improvement on the bound due to $\dim(G_m\square P_n)$ if G_m is not a complete graph.

Proposition 3.26. *Let G_m be any connected graph of order $m \geq 3$ and let P_n be the path of order $n \geq 2$. Then $2 \leq \zeta(G_m\square P_n) \leq m - 1$.*

Since $\zeta(K_m) = m - 1$ by Proposition 3.8, the next result follows by applying Theorem 3.20:

Corollary 3.27. *Let K_m be the complete graph of order $m \geq 3$ and let P_n be the path of order $n \geq 2$. Then $\zeta(K_m\square P_n) = m - 1$.*

If $m = 1$, then $\zeta(G_m\square P_n) = \zeta(P_n) = 1$. Also if $n = 1$, then $G_m\square P_n = G_m$. If $G_m = P_m$, then $\zeta(G_m\square P_n) = \zeta(G_{m,n}) = 2$ by Theorem 3.15 and therefore the lower bound in Theorem 3.26 is tight. Note that if G_m is connected and $m = 2$, then $G_2 = P_2$. The upper bound in Theorem 3.26 is tight by Corollary 3.27.

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3.4 Doubly resolving number

Since the doubly resolving number is a bound on the localization number, the doubly resolving number will now be investigated for certain graph classes. Lemma 3.7 states that $\dim(G) = m-1$ if and only if $G = K_m$. The following result shows that this is not true for doubly resolving sets:

Proposition 3.28. *For the complete bipartite graph $K_{1,m-1}$ of order $m \geq 3$, $\psi(K_{1,m-1}) = m-1$.*

Proof. Say $K_{1,m-1}$ has vertex set $V = \{v\} \cup U$ such that $d(v,u) = 1$ for all $u \in U$. Let $W \subseteq V$ be a set of vertices such that $|W| = m-2$ and let $X = V \setminus W$ contain the two vertices not in W. We consider two cases for X: $X \subseteq U$ and $v, u_i \in X$.

First assume $X \subseteq U$ and let $u_i, u_j \in X$. Then

$$d(u_i, v) - d(u_j, v) = 1 - 1 = 0$$

as well as

$$d(u_i, u_k) - d(u_j, u_k) = 2 - 2 = 0$$

for $u_k \in U$ such that W is not a doubly resolving set. Next, assume $v, u_i \in X$. Then $d(v, u_k) - d(u_i, u_k) = 1-2 = -1$ for any vertex $u_k \in U$ such that W is not a doubly resolving set. Therefore W is not a doubly resolving set of $V(G)$ such that $\psi(K_{1,m-1}) > m-2$. □

The next result shows that $\psi(G) \leq m-2$ for connected graphs of diameter at least three:

Proposition 3.29. *Let G be any connected graph of order m with a diameter of at least three. Then $\psi(G) \leq m-2$.*

Proof. Assume $\operatorname{diam}(G) = d$ and let $d(a,b) = d$ for $a, b \in V(G)$. Let $(a = v_0, v_1, \ldots, v_d = b)$ be a $a-b$ geodesic in G. Let $X = \{v_1, v_2\}$ and let $W = V(G) \setminus X$ such that $a, b \in W$ and $|W| = m-2$. Then the following two equations hold:

$$d(v_1, a) - d(v_2, a) = 1 - 2 = -1$$

and

$$d(v_1, b) - d(v_2, b) = (d-1) - (d-2) = 1$$

and thus W doubly resolves X. Next we consider vertex pairs where the one vertex is in W and the other in X. To this end, consider the vertex pair $\{v_i, v_j\}$ where $i \in \{1,2\}$ and $j \in \{0, 3, 4, \ldots, d\}$. First, let $j = 0$ such that $v_j = a$. Then $d(v_i, a) - d(a,a) = d(v_i, a) = i$ and $d(v_i, b) - d(a,b) = d - i - d = -i$ such that the vertex pair $\{v_i, a\}$ is doubly resolved by $a, b \in W$. Next, let $j \geq 3$. Then $d(v_i, v_j) - d(v_j, v_j) = j - i$ and $d(v_i, a) - d(v_j, a) = i - j$ such that $\{v_i, v_j\}$ is doubly resolved by $v_j, a \in W$. Therefore all vertex pairs in $V(G)$ are doubly resolved by W. □

3.5 Chapter summary

The chapter started with the calculation of $\zeta(K_{2,3})$. Thereafter some basic results on $\zeta(G)$ were proved, some of which are mentioned in literature without proof. In the second section, we looked at the localization number of complete graphs, cycles and grids.

The focus fell on Cartesian products in Section 3.3. Here two of the main results in the thesis were proved: a lower and upper bound for $\zeta(G \Box H)$. Specifically we showed that $\max\{\zeta(G), \zeta(H)\} \leq \zeta(G \Box H) \leq \zeta(G) + \psi(H) - 1$, where $\psi(H)$ is the doubly resolving number of H as introduced in [12]. This section was ended by showing that $\zeta(G_m \Box P_n) \leq m - 1$.

In the final section it was shown that, different to the dimension of a graph, there exist graphs other than K_n for which the doubly resolving number is equal to $n-1$.

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CHAPTER 4

Products with K_m

In this chapter the localization number of the Cartesian product of an arbitrary graph and the complete graph is investigated. Two special cases are considered in Sections 4.2 and 4.3, that of the Cartesian product of complete graphs, and the Cartesian product of an complete graph and a cycle.

4.1 General products with K_m

Let K_m be a complete graph of order m and G_n a graph of order n, where $m \geq n \geq 4$. From Theorems 3.20 and 3.24 we have that $m-1 \leq \zeta(K_m \Box G_n) \leq m+n-3$ since $\zeta(K_m) = m-1$ by Proposition 3.8 and $\psi(G_n) \leq n-1$. The following proposition provides an upper bound for $\zeta(K_m \Box G_m)$ and so doing improves the bound $\zeta(K_m \Box G_m) \leq 2m-3$:

Proposition 4.1. *Let G_m be any graph of order $m \geq 4$ and K_m be a complete graph of the same order. Then $\zeta(K_m \Box G_m) \leq m$.*

Proof. The Cop probes $B_1 = \{v_{0,0}, v_{1,1}, \ldots, v_{m-2,m-2}\} \cup \{v_{1,0}\}$ in the first turn, i.e., there is a probed vertex in all but one column and all but one row. Since $\zeta(G_m) \leq m-1$, it follows from Corollary 3.23 that no two vertices in the same safe set are in the same row or column. Let $b_0 = v_{0,0}, b_1 = v_{1,1}, \ldots, b_{m-2} = v_{m-2,m-2}$ and $b_{m-1} = v_{1,0}$. If an unprobed vertex x is in the same row as a probed vertex b_i, then $d(b_i, x) = 1$ since every row of $K_m \Box G_m$ is a copy of K_m. If x is in the same column as b_i, then $d(b_i, x) \geq 1$. If neither holds, then $d(b_i, x) > 1$. Vertex $v_{m-1,m-1}$ is the only vertex not in the same row or column as any probed vertex and therefore the only vertex where every entry of $\vec{D}(B_1, v_{m-1,m-1})$ is at least two. It follows that $v_{m-1,m-1}$ is not a safe vertex.

Consider unprobed vertex $x = v_{i,j}$ in column i and row j where $x \neq v_{m-1,m-1}$ and let $y \in V(G)$ such that $\vec{D}(B_1, x) = \vec{D}(B_1, y)$. First, assume $j < m-1$. Then x is in the same row as at least one probed vertex $b_j = v_{j,j}$. Since x and y are in the same safe set, we know that $d(b_j, x) = d(b_j, y) = 1$. This means that y is either in row j or column j. Since no two vertices in the same row are part of the same safe set, y is in column j. It therefore follows that all safe sets contain only two vertices.

If $j = m-1$, then $i \neq m-1$, i.e., x is in the same column as at least one probed vertex b_i. If $d(x, b_i) = 1$, then we have the same case as when $j < m-1$. Thus assume $d(x, b_i) \geq 2$ such that $d(y, b_i) \geq 2$. It follows that y is neither in row i nor in row $j = m-1$, but there exists a $k \leq m-2$ such that y is in row $k \neq i, j$. Then $d(y, b_k) = 1$ and $d(x, b_k) \geq 2$, a contradiction. Thus if $x = v_{i,m-1}$, then $\vec{D}(B_1, x) = \vec{D}(B_1, y)$ only if $d(x, b_i) = 1$.

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We now know that all safe sets are safe pairs. We now show that rows and columns 0 and 1 contain no safe vertices:

Case 4.1.1 Row 0. Say $j = 0$ such that x is in row 0. There are two probes in this row such that $d(x, b_0) = d(x, b_{m-1}) = 1$. But $d(y, b_0) = d(y, b_{m-1}) = 1$ only if y is in row 0, which is impossible. Thus no safe vertices exist in row 0.

Case 4.1.2 Column 0. Assume $x = v_{0,j}$ and $y = v_{k,l}$ where $k \neq 0$ and $l \neq j$. First, let $k = 1$. Then $d(y, b_0) = d_G(g_0, g_l) + 1$ and $d(y, b_{m-1}) = d_G(g_0, g_l)$, where $d(x, b_0) = d_G(g_0, g_j)$ and $d(x, b_{m-1}) = d_G(g_0, g_j) + 1$. Thus if $d(x, b_0) = d(y, b_0)$, then $d_G(g_0, g_j) = d_G(g_0, g_l) + 1$ such that $d(x, b_{m-1}) = d_G(g_0, g_l) + 2 \neq d(y, b_{m-1})$. Now, assume $k \geq 2$. Thus $d(y, b_0) = d_G(g_0, g_l) + 1 = d(y, b_{m-1})$. However $d(x, b_0) = d_G(g_0, g_j)$ and $d(x, b_{m-1}) = d_G(g_0, g_j) + 1$. Thus column 0 does not contain safe vertices.

Case 4.1.3 Column 1. Assume that $x = v_{1,j}$ and $y = v_{k,l}$ where $k \geq 2$ and $l \neq 0, j$. Then $d(y, b_0) = d_G(g_0, g_l)+1 = d(y, b_{m-1})$. However, $d(x, b_0) \neq d(x, b_{m-1})$ since $d(x, b_0) = d_G(g_0, g_j)+1$ and $d(x, b_{m-1}) = d_G(g_0, g_j)$. Therefore column 1 contains no safe vertices.

Case 4.1.4 Row 1. Assume that $x = v_{i,1}$ and $y = v_{k,l}$ where $k \neq 0, 1, i$ and $l \geq 2$. Then $d(x, b_1) = 1$ which implies that y is in column 1. From Case 4.1.3 it follows that row 1 contains no safe vertices.

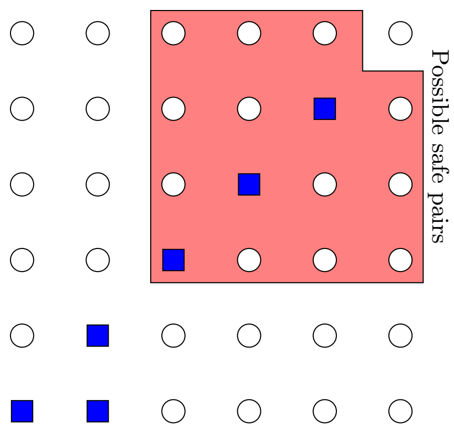

FIGURE 4.1: *The graph $K_6 \square G_6$ as an example to the probed vertices and safe sets in the proof of Proposition 4.1.*

As illustrated on $K_6 \square G_6$ in Figure 4.1, after probe B_1 safe pairs can only exist in rows and columns 2 to $m-1$, excluding vertex $v_{m-1,m-1}$. We thus assume the Robber is localized to $O_1 = \{v_{a,b}, v_{c,d}\}$ where $a, b, c, d \geq 2$ for $a \neq b, c$, $d \neq b, c$ and $v_{m-1,m-1}$ is not a safe vertex. In the second turn, the Robber can be at any vertex in columns a, c and rows b, d. The Cop now probes B_2 such that it contains $m-1$ rows and columns as well as including vertices $v_{a,b}, v_{c,d}$ and $v_{a,d}$. Similar to the probe of B_1, it can be shown that there are no safe vertices in rows b, d and columns a, c. Therefore, every vertex in $N[O_1]$ is resolved by B_2 such that the Cop wins in the second turn. $\square$

Proposition 4.2. *Let K_m be a complete graph of order m and G_n be any connected graph of order n such that $m > n \geq 4$. Then $\zeta(K_m \square G_n) = m - 1$.*

Proof. By Theorem 3.20 we have $\zeta(K_m \square G_n) \geq m - 1$, since $\zeta(K_m) = m - 1$ and $\zeta(G_n) \leq n - 1$ where $n < m$. Thus it only needs to be shown that the Cop can win using $m - 1$ cops. In the

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first turn, the Cop probes $B_1 = \{v_{0,0}, v_{1,1}, \ldots, v_{m-2,m-2}\}$, where row labels are taken modulo n. Say vertices x and y are part of the same safe set after probe B_1. Then both vertices are adjacent to at least one vertex in B_1, since B_1 has a vertex in every row. This implies that all possible safe sets have the form $\{v_{i,j}, v_{j,i}\}$. Probe B_1 and safe sets are illustrated on $K_7 \Box G_4$ in Figure 4.2.

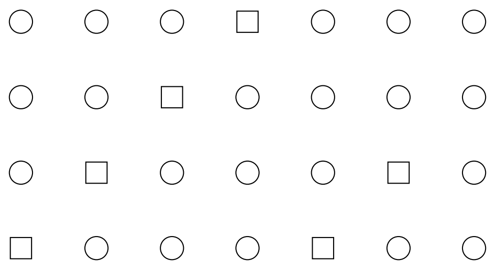

FIGURE 4.2: *The probe B_1 for $K_7 \Box G_4$.*

The Robber can now be at any vertex in columns and rows i and j. The Cop now probes B_2 such that it contains $m-1$ columns and $n-2$ rows. Further, B_2 must contain two vertices in row i, the one being $v_{i,j}$, and two vertices in row j, the one being $v_{j,i}$. By Corollary 3.23, no two vertices in the same row can belong to the same safe set. Further, every unprobed vertex in row i is adjacent to two probed vertices in row i and thus row i contains no safe vertices. Similarly, row j contains no safe vertices. Thus only vertices in columns i and j can be safe vertices. Say vertex $v_{i,\alpha}$ is a safe vertex and in the same safe set as vertex $v_{j,\beta}$ where $\alpha, \beta \neq i, j$. Let the two probes in row j be denoted by $v_{i,j}$ and $v_{l,j}$ and say $d(v_{i,\alpha}, v_{i,j}) = d$ such that $d(v_{i,\alpha}, v_{l,j}) = d+1$. Since $d(v_{i,\beta}, v_{i,j}) = d(v_{i,\beta}, v_{l,j})$, vertex $v_{j,\beta}$ is not in the same safe set as $v_{i,\alpha}$. It follows that vertex $v_{i,\alpha}$ can only be in the same safe set as a vertex in column i. By Corollary 3.23, such a safe set can only contain the two vertices in the two unprobed rows. The same is true for safe sets in column j.

Without loss of generality, assume the Robber is localized to robber set $O_2 = \{v_{i,\gamma_1}, v_{i,\gamma_2}\}$. In the next turn, the Robber can be at any vertices in column i as well as rows γ_1 and γ_2. The Cop now probes B_3 such that $m-1$ columns are probed, $n-1$ rows are probed, column i is unprobed, row γ_1 contains two probes and row γ_2 contains a probe. As before, row γ_1 can contain no safe vertices. Say some vertex v_{κ_1,γ_2} is a safe vertex and vertex v_{κ_2,γ_2} is the probe in this row. Then $d(v_{\kappa_1,\gamma_2}, v_{\kappa_2,\gamma_2}) = 1$ such that any vertex in the same safe set as v_{κ_1,γ_2}, must be in column κ_2. However, since $\kappa_2 \neq i$, the only vertex in $N[O_2]$ in column κ_2, is in row γ_1. It follows that row γ_2 contains no safe vertices and thus safe vertices can only be in column i. By Corollary 3.23 this is not possible and therefore $N[O_2]$ is resolved by B_3. □

Note that Proposition 4.2 only gives an upper bound to $\zeta(K_m \Box G_n)$ if $m > n$, where Proposition 4.1 only handles the case when $m = n$. It therefore remains to give an upper bound to $\zeta(K_m \Box G_n)$ for $n > m$:

Proposition 4.3. *Let K_m be a complete graph of order m and G_n be any connected graph of order n such that $n > m \geq 4$. Then $\zeta(K_m \Box G_n) \leq n-1$.*

Proof. In the first turn, the Cop probes $B_1 = \{v_{0,0}\} \cup \{v_{0,1}, v_{1,2}, \ldots, v_{n-3,n-2}\}$ such that every column is probed and every row except row $n-1$. Thus it follows from Corollary 3.23 that no two vertices in the same row or column can belong to the same safe set and that a safe set will only contain two vertices. If $G_n = K_n$, it follows from Proposition 4.2 that $\zeta(K_m \Box G_n) = n-1$ and hence we assume that $G_n \neq K_n$ such that there exists at least one pair of non-adjacent vertices

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in G_n. Say that after the first turn, the Robber is localized to robber set $O_1 = \{v_{x,y}, v_{w,z}\}$ where $x \neq w$ and $y \neq z$. Let $l = w - 1$, unless $w - 1 = x$ in which case we define $l = x - 1$. Note that row and column indices are taken modulo n and modulo m respectively. In the second turn, the Robber can be at any vertices in rows y and z as well as its neighbours in columns x and w. For the Cop's second probe, B_2 will be chosen such that every column and row is probed except one column and one row. Thus, the safe sets contain two vertices in unique rows and columns. Label the first two probed vertices $b'_1 = v_{l,y}$ and $b'_2 = v_{l,z}$. Assume row y contains a safe vertex v. Since b'_1 is adjacent to v, the other safe vertex will be in column l. The only other vertex in $N[O_1]$ in column l, is vertex $v_{l,z}$. Since $b'_2 = v_{l,w}$, row y does not contain safe vertices. The same can be shown for row z.

We say that a vertex is incident to a non-existent edge if the vertex is not a universal vertex. Note that since we assume $G_n \neq K_n$, there is at least one such vertex. The remaining $m - 3$ vertices in B_2 depend on whether a vertex in the robber set O_1 is incident to a non-existent edge or not.

Strategy 4.3.1 The robber set is incident to a non-existent edge in G_n. Assume vertices $v_{w,k}$ and $v_{w,z}$ are not adjacent in $G \Box H$ such that g_k and g_z are not adjacent in G_n. The unprobed column in B_2 will be column x and the unprobed row will be row k. Probes $b'_3, b'_4, \ldots, b'_{m-1}$ are chosen to be in unique rows and columns, omitting columns x and l as well as rows y, z and k. The remaining $n - m$ vertices are chosen to be in column w and the unused rows, omitting row k. This is illustrated on $K_4 \Box G_6$ in Figure 4.3.

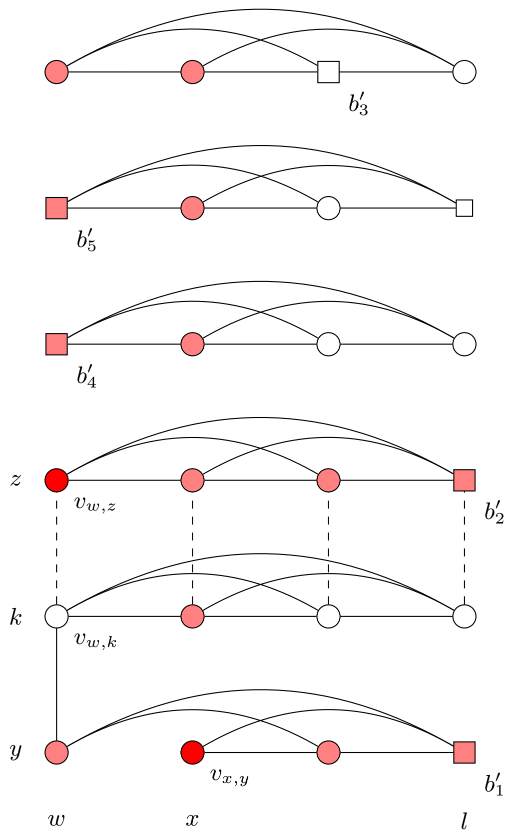

FIGURE 4.3: *The graph $K_4 \Box G_6$ as an example to Strategy 4.3.1.*

Now, say probe b'_3 is in row e and column f. There are potentially two vertices in this column in $N[O_1]$: $v_{w,e}$ and $v_{x,e}$. If one of the vertices is a safe vertex, the other safe vertex will be in

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column f. However, the only vertices in $N[O_1]$ in column f, are in rows y and z, which do not contain safe vertices and therefore row e does not contain a safe vertex. Similarly, we can show that none of the columns of probes b'_3 to b'_{m-1} contain safe vertices.

Therefore if v is in a safe set, then v lies in column x or w, and in row k or the rows of probes $b'_m, b'_{m+1}, \ldots, b'_{n-1}$. Since no safe set is contained in a single column, each of column x or w contains a safe vertex. Since probes b'_m to b'_{n-1} are in column w, the safe vertex in column w can only be vertex $v_{w,k}$. However, this vertex is not in $N[O_1]$ since $v_{w,k}$ is not adjacent to $v_{w,z}$ in $G\Box H$. Therefore no safe vertices exist after the Cop probes B_2 and the Cop wins.

Strategy 4.3.2 The robber set is not incident to a non-existent edge in G_n. Two cases are considered: $n-1 = m$ and $n-1 > m$.

Case 4.3.1 $n-1=m$. Say G_n vertices g_a and g_b are not adjacent, where $a, b \neq y, z$. In this case, we probe every column except column w and every row except row b. The Cop chooses vertices $b'_3, b'_4, \ldots, b'_{m-1}$ to be in unique rows, but omitting columns x, w and l as well as rows a and b. Again the rows of these probes contain no safe vertices. Let $b'_m = v_{x,a}$. Probe B_2 now covers $m-1$ columns and $n-1$ rows. This is illustrated for $K_4\Box G_5$ on the left of Figure 4.4. Now, if a safe set exists after probe B_2, the one vertex in the safe set will be $v_{x,b}$. The other vertex will be vertex $v_{w,a}$. This is not possible, since b'_m is not adjacent to $v_{x,b}$ and therefore the Cop wins.

Note that this strategy would not work if $n-1 > m$, since vertex $v_{w,a}$ need not be the other safe vertex.

Case 4.3.2 $n-1>m$. In this case, the Cop chooses vertices $b'_3, b'_4, \ldots, b'_{m-1}$ to be in unique rows and columns, omitting columns l, x and w as well as rows y and z. All the remaining $n-m$ vertices are chosen to be in column w. This probe is illustrated for $K_4\Box G_6$ on the right of Figure 4.4.

Next we consider possible safe sets after probing B_2. Say $b'_3 = v_{c,d}$. The vertices in $N[O_1]$ in row d are $v_{w,d}$ and $v_{x,d}$. If one of these vertices belongs to a safe set, the other vertex in the safe set will be in column c. However, the only two vertices in this column that are in $N[O_1]$, are in rows y and z. Both these rows do not contain safe vertices and therefore row d also does not contain safe vertices. This argument can be repeated for probes $b'_4, b'_5, \ldots, b'_{n-1}$. Say row t is the unprobed row. The only possible safe vertices now are in the row t as well as the columns of probes $b'_m, b'_{m+1}, \ldots, b'_{n-1}$. If a safe set exists, it will contain two vertices where the one vertex is in column w and the other in column x. The last $n-m$ probes are all in column w and therefore one of the safe vertices in the safe set will be the vertex $v_{w,t}$. The other vertex in the safe set, say u, will be in column x and in one of the rows of the last $n-m$ probes. In each of these cases, vertex $v_{w,t}$ is in the same row as at least two of the probes $b'_m, b'_{m+1}, \ldots, b'_{n-1}$. However, only one of these two probes is adjacent to u. Therefore the other probe cannot be a neighbour of $v_{w,t}$ such that this edge does not exist in G_n. Thus in the next turn, $v_{w,t} \in O_2$ is incident with a non-existent edge and the Cop can use Strategy 4.3.1 in the next turn to win.

This completes the Cop's strategies and therefore the Cop can win on $K_m\Box G_n$ for $n > m$ using $n-1$ cops. $\Box$

From Propositions 4.1 and 4.3 we have the following upper bound on the Cartesian product of any graph and a complete graph:

Theorem 4.4. *Let G_m be any connected graph of order $m \geq 4$ and K_n be the complete graph of order $n \geq 4$. If $m = n$, then $\zeta(G_m\Box K_n) \leq m$. If not, then $\zeta(G_m\Box K_n) \leq \max\{m, n\} - 1$. Both these bounds are tight if $G_m = K_m$.*

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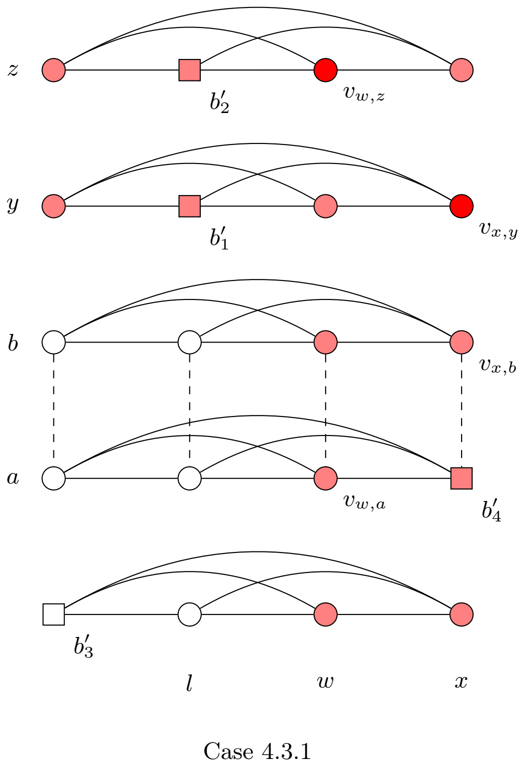

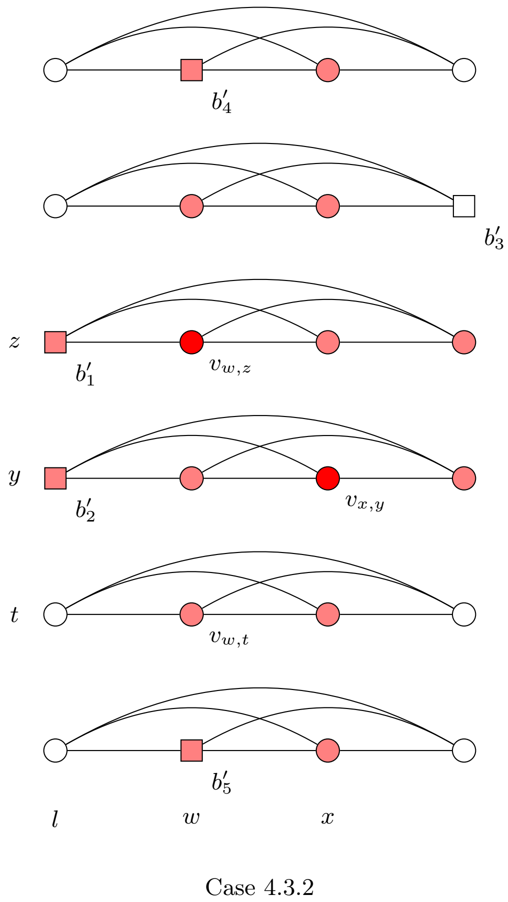

FIGURE 4.4: *On the left, $K_4\Box G_5$ as an illustration to Case 4.3.1. On the right, $K_4\Box G_6$ as an illustration to Case 4.3.2.*

4.2 The product $K_m\Box K_n$

Proposition 4.5. *Let K_m be a complete graph of order $m \geq 4$. Then $\zeta(K_m\Box K_m) > m-1$.*

Proof. It needs to be shown that the Robber always has a winning strategy if the Cop plays with $m-1$ cops. To this end, say the Cop probes $B_1 = \{b_1, b_2, \ldots, b_{m-1}\}$ in the first turn. Let $v \in (V(K_m\Box K_m) \setminus B_1)$ and consider $d_i = d(b_i, v)$ for $i = 1, 2, \ldots, m-1$. Then $d_i \in \{1, 2\}$, where $d_i = 1$ if v is in the same row or column as b_i and $d_i = 2$ otherwise.

If each b_i is in a different row and column, let v_1 be the vertex in the same column as b_x and the same row as b_y for distinct vertices $b_x, b_y \in B_1$. Thus $d(b_x, v_1) = d(b_y, v_1) = 1$ and $d(b_i, v_1) = 2$ for $i \neq x, y$. Similarly, there exists a vertex $v_2 \in V$ that is in the same row as b_x and the same column as b_y such that $\vec{D}(B_1, v_1) = \vec{D}(B_1, v_2)$. Hence $\{v_1, v_2\}$ is a safe set.

Now assume that there exist at least one row (or column) with more than one vertex of B_1. The set B_1 is therefore contained in at most $m-2$ rows such that the projection of B_2 onto any column of $K_m\Box K_m$ contains only $m-2$ vertices. Since $\zeta(K_m) = m-1$, it follows from Corollary 3.23 that there exists a safe set in every column of $K_m\Box K_m$. It follows that the Robber is localized to either a safe pair containing two vertices in the same column (or row), or a safe pair containing $v_{i,j}$ and $v_{j,i}$ for some $i, j \leq m-1$. Hence, at the start of the second turn, the Robber can either be at two rows or two columns (or both). Without loss of generality assume the Robber can be at two columns c_1 and c_2.

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Say the Cop probes $B_2 = \{b_1', b_2', \dots, b_{m-1}'\}$ in the second turn. If B_2 is not contained in $m-1$ rows, then it follows from Lemma 3.22 that $N[O_1]$ contains a safe set. Now assume all vertices b_j' are in different rows and columns for $j = \{1, 2, \dots, m-1\}$ such that one column is unprobed. There are now two cases two consider: when the unprobed column is either column c_1 or c_2 and when it is not.

Case 4.5.1 Unprobed column c_1. Results follow similarly if c_2 is the unprobed column. Say the probe in column c_2 is $b_1' = v_{c_2,y}$. If row p is the unprobed row then $D(B_2, v_{c_1,y}) = D(B_2, v_{c_2,p}) = [1, 2, 2, \dots, 2]$ such that a safe pair always exists.

Case 4.5.2 Columns c_1 and c_2 are probed. Say $b_1' = v_{c_1,y}$ and $b_2' = v_{c_2,z}$. Then vertices $v_{c_1,z}$ and $v_{c_2,y}$ are adjacent to both b_1 and b_2, but not adjacent to any other probed vertices. Thus $\vec{D}(B_2, v_{c_1,z}) = \vec{D}(B_2, v_{c_2,y}) = [1, 1, 2, 2, \dots, 2]$ such that a safe pair exists.

Therefore irrespective of the robber set O_1, a safe pair will exist in $N[O_1]$ after the Cop's second probe and the Robber can continuously avoid capture if $m-1$ cops are used. $\Box$

Propositions 4.1, 4.2 and 4.5 prove the following theorem:

Theorem 4.6. *Let $K_m \Box K_n$ be the product of two complete graphs with orders $m, n \geq 4$. If $m > n$, then $\zeta(K_m \Box K_n) = m - 1$. If $m = n$, then $\zeta(K_m \Box K_n) = m$.*

4.3 The product $K_m \Box C_n$

Let $m \geq 4$. By [12] we know that $\psi(K_m) = m-1$ for $m \geq 3$, where Proposition 3.8 and Theorem 3.13 state that $\zeta(K_m) = m-1$ and $\zeta(C_n) = 1$ for $n \geq 7$ respectively. Therefore by Theorems 3.20 and 3.24, $\zeta(K_m \Box C_n) = m-1$ for $m \geq 3$ and $n \geq 7$. In order to further calculate $\zeta(K_m \Box C_n)$ for $n \leq 6$, the following results by Cáceres et al. will be used:

Proposition 4.7 [12]. *For all $m \geq 4$ and $n \geq 3$, the following holds:*

$$\dim(K_m \Box C_n) = \begin{cases} 3 & \text{if } m = 4 \text{ and } n \text{ is even,} \\ 4 & \text{if } m = 4 \text{ and } n \text{ is odd, and} \\ m-1 & \text{if } m \geq 5. \end{cases}$$

By Theorem 3.20 and Proposition 4.7, we have the following corollary:

Corollary 4.8. *Let K_m be a complete graph of order $m \geq 4$ and C_n a cycle of order $n \geq 3$. Then*

$$\zeta(K_m \Box C_n) = \begin{cases} m-1 & \text{if } m \geq 5, \\ m-1 & \text{if } m = 4 \text{ and } n \text{ is even, and} \\ \in \{3, 4\} & \text{if } m = 4 \text{ and } n \text{ is odd.} \end{cases}$$

Therefore $\zeta(K_m \Box C_n)$ has been determined for all values of $m \geq 1$ and $n \geq 3$ except for $m = 4$ while $n \in \{3, 5\}$.

Proposition 4.9. *Let K_4 be the complete graph of order four and C_5 be the cycle of order five. Then $\zeta(K_4 \Box C_5) = 3$.*

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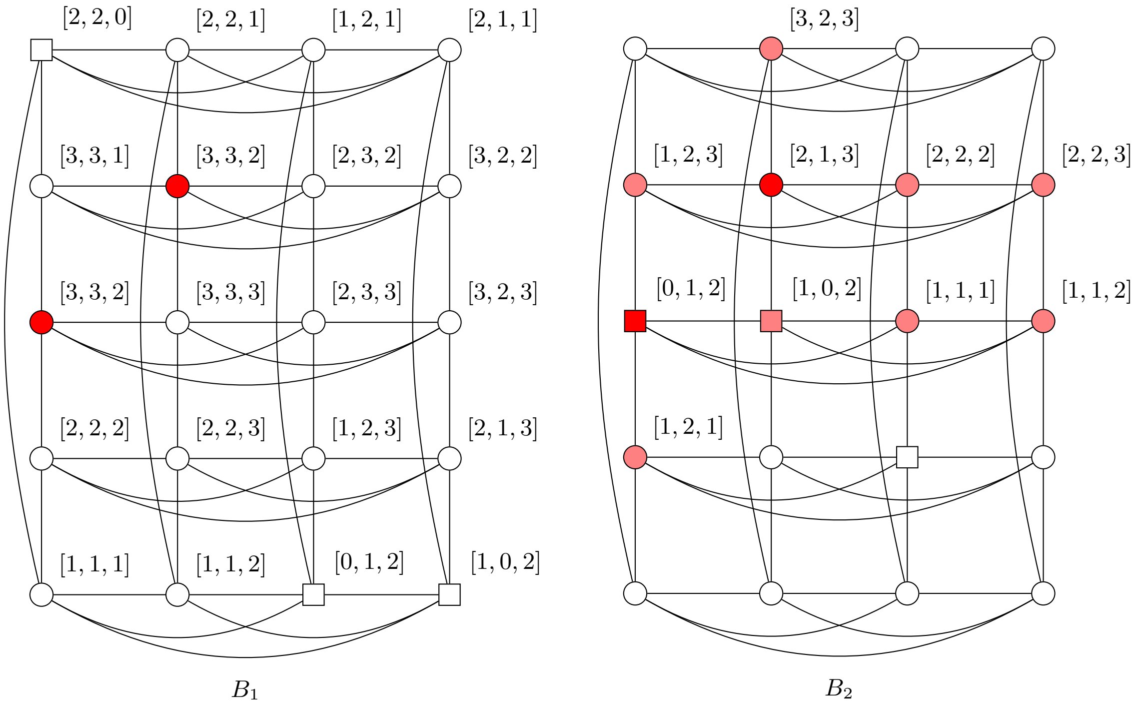

FIGURE 4.5: *The probes of Proposition 4.9. Probe B_1 on the left and B_2 on the right.*

Proof. In the first turn, the Cop probes $B_1 = \{v_{2,0}, v_{3,0}, v_{0,4}\}$. The distances from vertices to B_1 are given on the left in Figure 4.5, where the only safe set is indicated in red.

Thus after the first turn, the Robber is localized to robber set $O_1 = \{v_{0,2}, v_{1,3}\}$. The Cop now probes $B_2 = \{v_{0,2}, v_{1,2}, v_{2,1}\}$. The probe B_2 with distances from vertices in $N[O_1]$ to B_1 are given on the right in Figure 4.5, where it can be seen that the Cop wins. □

Proposition 4.10. *Let K_4 be the complete graph of order four and C_3 be the cycle of order three. Then $\zeta(K_4 \Box C_3) = 3$.*

Proof. The Cop probes $B_1 = \{v_{0,0}, v_{1,0}, v_{1,2}\}$ in the first turn. The distances from vertices to probe B_1 are given on the left in Figure 4.6, where it can be seen that all safe sets have the form $\{v_{2,j}, v_{3,j}\}$ for $j \in \{0, 1, 2\}$.

If the Robber is not in columns $i = 0$ or $i = 1$, the Cop requires a second probe and the Robber is localized to some robber set $O = \{v_{2,j}, v_{3,j}\}$. The Cop now probes $B_2 = \{v_{2,j}, v_{3,j}, v_{0,j-1}\}$. Say $j = 0$ such that the distances from vertices to probe B_2 are given on the right in Figure 4.6, where it can be seen that the Cop wins. In a similar way it can be shown that the Cop also wins if $j = 1$ and $j = 2$. □

The localization number of $K_m \Box C_n$ has thus been determined, and is given in the following theorem:

Theorem 4.11. *Let K_m be a complete graph of order $m \geq 4$ and C_n a cycle of order $n \geq 3$. Then $\zeta(K_m \Box C_n) = m - 1$.*

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4.4 Chapter summary

In this chapter, we considered the product $K_m \square G_n$. We started by showing that $\zeta(K_m \square G_m) \leq m$, where $\zeta(K_m \square G_n) = m - 1$ for $m > n$. In doing so, $\zeta(K_m \square K_n)$ was calculated for all m and n. It was also established that $\zeta(G_m \square K_n) \leq m - 1$ if $m > n$.

The chapter concluded with the case when $G_m = C_m$ and it was shown that $\zeta(K_m \square C_n) = m - 1$.

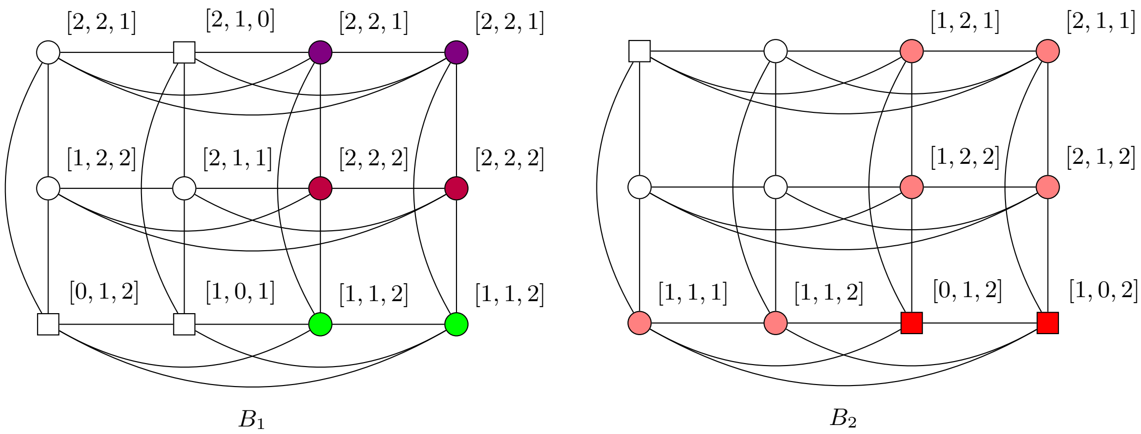

FIGURE 4.6: *The probes of Proposition 4.10. Probe B_1 on the left and B_2 on the right.*

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CHAPTER 5

Products with C_n

In this chapter the localization number of the Cartesian product of a cycle and a graph of order m is considered. First $\zeta(C_m \square C_n)$ is calculated by considering three cases of $C_m \square C_n$: odd by odd, odd by even and even by even. In Section 5.1 the Cartesian product of two odd cycles are considered. After that the product of an odd and even cycle is considered in Section 5.2 and lastly the product of two even cycles are considered in Section 5.3. In each of the sections we will first consider the case where m, n is big enough, after which the special cases will be considered. Then in Section 5.4 we calculate $\zeta(P_m \square C_n)$ and close off by giving an upper bound to $\zeta(G_m \square C_n)$.

Definition 5.1 Second difference. For probe $B = \{b_1, b_2\}$, vertex v and distance vector $\vec{D}(B, v) = [a, b]$, we define the second difference DD as $DD(B, v) = b - a$.

Clearly, if $DD(B, x) \neq DD(B, y)$, then $\vec{D}(B, x) \neq \vec{D}(B, y)$ for any two vertices x, y. For each second difference that is not unique to a single vertex, there exists a set of vertices where the Robber is potentially safe. This set will be called a safe house.

Definition 5.2 Safe house. For a graph G with probe B, a safe house S_h is the set of all vertices $v \in V(G)$ such that $DD(B, v) = h$. Note that safe sets are confined to a specific safe house.

Definition 5.3 Cop house. Let G be a graph where the Cop probes B_α in turn α. A cop house is a subset of $V(G)$ that contains only vertices from different safe sets.

A cop house is therefore “locally unique”: if the Robber is restricted to movement in a cop house in turn α, the Cop wins immediately. Note that a cop house may contain safe vertices, but all vertices in a cop house belong to different safe sets.

Definition 5.4 Diagonal safe pair. A diagonal safe pair is a safe set that contains two safe vertices that can be written as $\{v_{a,b}, v_{a+1,b+1}\}$ (positive diagonal) or $\{v_{a,b}, v_{a+1,b-1}\}$ (negative diagonal) for integers a and b.

Definition 5.5 Horizontal safe pair. A horizontal safe pair S_d^h is a safe set that contains two safe vertices a distance of d apart that can be written as $\{v_{a,b}, v_{a+d,b}\}$.

Definition 5.6 Vertical safe pair. A vertical safe pair S_d^v is a safe set that contains two safe vertices a distance of d apart that can be written as $\{v_{a,b}, v_{a,b+d}\}$.

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Let $C_m \square C_n$ be the Cartesian product of two cycles of order m and n respectively. Vertices will be labeled the same as in grids, where the indices i and j of $v_{i,j}$ are $\bmod m$ and $\bmod n$ respectively. The main result in this chapter is the following theorem:

Theorem 5.1. *Let $C_m \square C_n$ be a product of cycles with m, n integers such that $m \geq n \geq 3$. If $m = n = 3$ or if m is even while $n = 4$, then $\zeta(C_m \square C_n) = 3$. Otherwise, $\zeta(C_m \square C_n) = 2$.*

From Theorem 3.24 we have the following result for cycles:

$$\zeta(C_m \square C_n) \leq \zeta(C_m) + \psi(C_n) - 1. \tag{5.1}$$

From [9] and [24] it follows that

$$\zeta(C_m) = \begin{cases} 1 & \text{for } m \geq 7 \\ 2 & \text{for } m \leq 6. \end{cases}$$

Further, Cáceres et al. [12] proved the following:

Lemma 5.2 ([12]). *Let C_n be a cycle of order $n \geq 3$. Then*

$$\psi(C_n) = \begin{cases} 2 & \text{for odd } n \\ 3 & \text{for even } n. \end{cases}$$

It follows that $\zeta(C_m \square C_n) = 2$ for $m \geq 7$ and n odd. The value of $\zeta(C_m \square C_n)$ for $m \leq 6$ or when n is even will be determined in three separate cases: the product of two odd cycles, an even and an odd cycle and lastly two even cycles.

5.1 Odd by odd

First consider the localization number of $C_m \square C_n$ where m and n are odd and $m \geq n$. Since n is odd, it is known that $\zeta(C_m \square C_n) = 2$ when $m \geq 7$ and therefore only two cases for m are considered here: $m = 3$ and $m = 5$. For $m = n = 3$, we prove the following result:

Proposition 5.3. *Let $C_3 \square C_3$ be a product of cycles. Then $\zeta(C_3 \square C_3) = 3$.*

Proof. Note that $\zeta(C_3) = \psi(C_3) = 2$ and therefore $\zeta(C_3 \square C_3) \leq 3$ by Equation (5.1). It follows that we only need to show that there exists a winning strategy for the Robber if only two cops are used. To this end, say the Cop probes $B_1 = \{b_1, b_2\}$ in the first turn and let

$$Z = V(C_3 \square C_3) \setminus B_1$$

be the vertices not probed by the Cop. Since $\text{diam}(C_3 \square C_3) = 2$, the distance vector $\vec{D}(B_1, z)$ for $z \in Z$ may be one of four unique distance vectors. Since $|Z| = 7$, there exists safe vertices and the Robber can avoid capture in the first turn. Say u and v are two vertices in the same safe set and the Robber is at one of these two vertices. Then $|N[\{u, v\}]| \geq 5$ and again by the pigeonhole principle, at least two vertices in $N[\{u, v\}]$ are not uniquely defined by B_2. Thus, at any turn, there are at least two vertices where the Robber is safe, irrespective of the Cop's probe, and therefore $\zeta(C_3 \square C_3) \geq 3$. $\square$

Now consider the case when $m = n = 5$.

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Proposition 5.4. *Let $C_5 \square C_5$ be the product of two cycles. Then $\zeta(C_5 \square C_5) = 2$.*

Proof. The Cop probes $B_1 = \{v_{2,4}, v_{2,2}\}$ in the first turn. For any vertex $v_{i,j}$, the distance vector $\vec{D}(B_1, v_{i,j})$ is given in Figure 5.1. Safe houses are indicated with the same colour and vertices that belong to the same safe set have the same shape and colour. The probed vertices are indicated as squares and empty vertices do not form part of a safe set. The distance from a vertex to B_1 is indicated above the vertex. From Figure 5.1 it can be seen that all safe sets have the form $\{v_{i,j}, v_{4-i,j}\}$ for $i, j = 0, 1, 2, 3, 4$. Also, notice the presence of a cop house in columns 0 to 2.

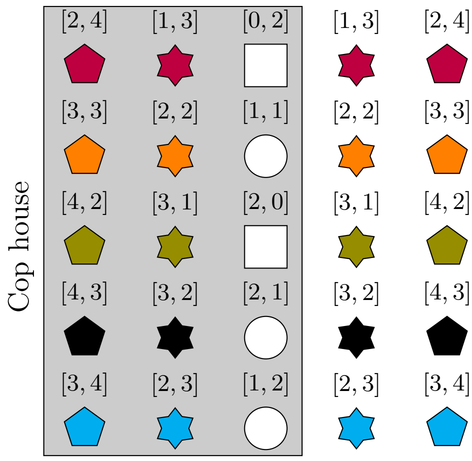

FIGURE 5.1: *The product $C_5 \square C_5$ where the safe sets, safe vertices and safe houses for probe B_1 are indicated. Safe houses are indicated with the same colour and vertices that belong to the same safe set have the same shape and colour. The probed vertices are indicated as squares and empty vertices do not form part of a safe set. The distance from a vertex to B_1 is indicated above the vertex.*

If the Robber was at a vertex in column 2, the Cop wins immediately. If not, the Robber is localized to the robber set $O_1 = \{v_{i,j}, v_{4-i,j}\}$ such that $N[O_1]$ is contained in rows $j-1$, j and $j+1$. For the second probe, the Cop probes $B_2 = \{v_{4,j+1}, v_{2,j+1}\}$ such that B_2 is a translation of B_1, rotated by 90 degrees. Thus probe B_2 creates a cop house in rows $j-1$ to $j+1$. Since $N[O_1]$ is contained in these rows, the Cop wins. □

The proof for Proposition 5.4 is modified slightly for $C_5 \square C_3$ by changing the first probe to $B_1' = \{v_{2,1}, v_{2,2}\}$ and keeping the second probe the same.

Proposition 5.5. *Let $C_5 \square C_3$ be the product of two cycles. Then $\zeta(C_5 \square C_3) = 2$.*

It follows that for m and n odd, $\zeta(C_m \square C_n) = 2$, unless $m = n = 3$.

5.2 Odd by even

Next, consider the case where m is odd and n is even. Since $\psi(C_m) = 2$, it follows that $\zeta(C_m \square C_n) = 2$ for $n \geq 8$. To determine $\zeta(C_m \square C_n)$ for $n \leq 6$ we start by determining the safe houses for the chosen probes.

Lemma 5.6. *Let $C_{2p+1} \square C_{2q}$ be a product of cycles with $p \geq 1$ and $q \in \{2, 3\}$. If the Cop probes $B_1 = \{v_{p,2q-1}, v_{p,q-1}\}$ in the first turn, all safe sets will be of the form $O = \{v_{i,j}, v_{2p-i,j}, v_{i,2q-2-j},$*

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$v_{2p-i,2q-2-j}\}$ for $i = 0, 1, \ldots, 2p$ and $j = 0, 1, \ldots, 2q-1$. Further for R_1 as the set of all vertices $v_{x,y}$ where $x = 0, 1, \ldots, p$ and $y = q-1, q, \ldots, 2q-1$, R_1 is a cop house.

Proof. Say the Cop probes

$$B_1 = \{v_{p,2q-1}, v_{p,q-1}\} \tag{5.2}$$

such that the distance vector $\vec{D}(B_1, v_{i,j})$ is given by

$$\vec{D}(B_1, v_{i,j}) = [|p-i| - |q-1-j| + q, |p-i| + |q-1-j|] \tag{5.3}$$

for any vertex $v_{i,j}$. This is illustrated on $C_7 \Box C_6$ in Figure 5.2.

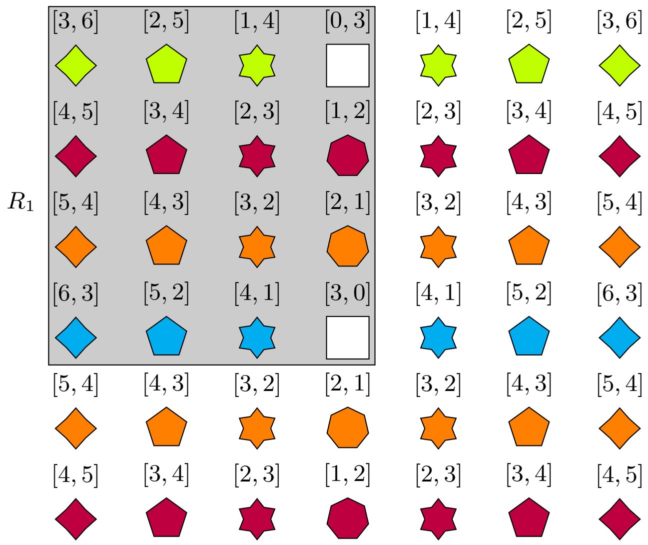

FIGURE 5.2: *The graph $C_7 \Box C_6$ with probe B_1 as in Equation (5.2). The distances from vertices to B_1 as well as the cop house R_1 are shown.*

The second difference is given by $DD(B_1, v_{i,j}) = 2|q-1-j| - q$ and is therefore not dependant on the column of vertex $v_{i,j}$ and only on its row. Consider two vertices v_{i_1,j_1}, v_{i_2,j_2} and let $DD(B_1, v_{i_1,j_1}) = DD(B_1, v_{i_2,j_2})$ such that

$$|q-1-j_1| = |q-1-j_2|. \tag{5.4}$$

There are two solutions to Equation (5.4): $j_1 = j_2$ and $j_1 + j_2 = 2q-2$. All vertices in the same row are therefore in the same safe house, where vertices in different rows are in the same safe house only if $j_1 + j_2 = 2q-2$. It follows that that every safe house contains two rows of $C_{2p+1} \Box C_{2q}$, except the two safe houses containing rows $q-1$ and $2q-1$ respectively. In order to calculate the safe sets, let $\vec{D}(B_1, v_{i_1,j_1}) = \vec{D}(B_1, v_{i_2,j_2})$. Since the two vertices are in the same safe house, it follows from Equations (5.3) and (5.4) that two vertices are in the same safe set if $|p-i_1| = |p-i_2|$. The only nontrivial solution is $i_1 + i_2 = 2p$. Therefore all safe sets have the form $O_1 = \{v_{i,j}, v_{2p-i,j}, v_{i,2q-2-j}, v_{2p-i,2q-2-j}\}$ for $i = 0, 1, \ldots, 2p$ and $j = 0, 1, \ldots, 2q-1$. Note that if $j = q-1$, $j = 2q-1$ or $i = p$, the safe set only contains two vertices.

Next consider R_1. By the solution to Equation (5.4), two vertices in R_1 only belong to the same safe house if they are in the same row. Therefore two vertices v_{x_1,y_1}, v_{x_2,y_2} in R_1 are only part of the same safe set if $|p-x_1| = |p-x_2|$ such that $x_1 + x_2 = 2p$. This is never true inside R_1 and therefore every two vertices in R_1 belong to different safe sets and R_1 is a cop house. □

Since the $C_m \Box C_n$ is vertex transitive, the following corollary follows:

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Corollary 5.7. *Say the Cop probes $B_2 = \{v_{a+p,b+q}, v_{a+p,b}\}$ in the second turn such that $B_2 = g(B_1)$, where g is a translation. Then for $R_2 = g(R_1)$, R_2 is a cop house. Further two distinct vertices v_{i_1,j_1} and v_{i_2,j_2} are only part of the same safe set if $(i_1 + i_2) \equiv 2a + 2p \bmod (2p+1)$ or $(j_1 + j_2) \equiv 2b \bmod (2q)$. Note that if $i_1 \neq i_2$ and $j_1 \neq j_2$, then both these equations need to hold.*

The lemma can also easily be adapted for the even by even case:

Corollary 5.8. *Let $C_{2p} \Box C_{2q}$ be a product of cycles with $p \geq q \geq 4$ and say the Cop probes $B_1 = \{v_{p,2q-1}, v_{p,q-1}\}$ in the first turn. Then all safe sets will be of the form $O = \{v_{i,j}, v_{2p-i,j}, v_{i,2q-2-j}, v_{2p-i,2q-2-j}\}$ for $i = 0, 1, \ldots, 2p$ and $j = 0, 1, \ldots, 2q-1$. Further, for R_1 defined as in Lemma 5.6, R_1 is a cop house. Also if f is a translation such that $B_2 = f(B_1)$, then $R_2 = f(R_1)$ is a cop house.*

Now for the case of m odd and $n \leq 6$ we omit the restriction that $m \geq n$. If this restriction is included, a separate proof will be needed for even by odd, which will be equivalent to the one given here.

Proposition 5.9. *Let $C_{2p+1} \Box C_{2q}$ be a product of cycles with $p \geq 1$ where $q \in \{2, 3\}$. Then $\zeta(C_{2p+1} \Box C_{2q}) = 2$.*

Proof. In the first turn, the Cop probes $B_1 = \{v_{p,2q-1}, v_{p,p-1}\}$ such that the Robber is localized to robber set

$$O_1 = \{v_{i,j}, v_{2p-i,j}, v_{i,2q-2-j}, v_{2p-i,2q-2-j}\} \tag{5.5}$$

by Lemma (5.6), where $i = 0, 1, \ldots, 2p$ and $j = 0, 1, \ldots, 2q-1$. Now define $d_i = d(v_{i,j}, v_{2p-i,j})$ and $d_j = d(v_{i,j}, v_{i,2q-2-j})$ such that the robber set can be written as $O_1 = \{v_{a,b}, v_{a+d_i,b}, v_{a,b+d_j}, v_{a+d_i,b+d_j}\}$ where $v_{i,j} = v_{a,b}$ need not be true. The distances d_i and d_j are given by $d_i = \min\{2i+1, 2p-2i\}$ and $d_j = \min\{2j+2, 2q-2-2j\}$. Note that d_i and d_j are also calculated modulo m and n respectively. It follows that $d_i \leq p$ and since $q \in \{2, 3\}$, we have that $d_j \in \{0, 2\}$. In the second turn, the Cop probes

$$B_2 = \{v_{a+p,b+q}, v_{a+p,b}\} \tag{5.6}$$

such that $B_2 = g(B_1)$ where g is some translation function. The vertices of O_1 will be labeled $v_{a,b} = u_1, v_{a,b+d_j} = u_2, v_{a+d_i,b+d_j} = u_3$ and $v_{a+d_i,b} = u_4$ with neighbours $u_l^N, u_l^E, u_l^S, u_l^W$ for $l = 1, 2, 3, 4$. Let R_2 be the set of all vertices $v_{s,t}$ where $s = a, a+1, \ldots, a+p$ and $t = b, b+1, \ldots, b+q$ such that $R_2 = g(R_1)$. Then R_2 is a cop house by Corollary 5.7. The vertices in $N[O_1]$ as well as probe B_2 are illustrated in Figure 5.3. In the figure, the region R_2 is indicated with a dotted square.

We now show that every safe set in the second turn is a vertical or horizontal safe pair, where the two vertices in the safe set are at distance one or two from each other. First, consider $N[\{u_1, u_4\}]$. By Corollary 5.7 all vertices in row b belong to a safe house and no vertices outside of this row are part of the same safe house. Further, safe sets in this row only contain two vertices and therefore none of the vertices in row b are part of vertical safe pairs. By Corollary 5.7 it follows that $\{u_1, u_1^W\}, \{u_1^N, u_1^S\}$ and $\{u_4^N, u_4^S\}$ are safe sets. Vertex u_1^E can only be in a safe set with a vertex outside R_2 in row b. Thus the only option is u_4^E if $d_i = p$. Then, $i_1 + i_2 = (a+1) + (a + d_i + 1) = 2a + p + 2$. This only satisfies $(i_1 + i_2) \equiv 2a + 2p \bmod (2p+1)$ if $p = 2$, in which case the safe set $\{u_1^E, u_4^E\} = S_2^h$. If $d_i = p > 2$, then $\{u_4^E, u_4^W\}$ is a horizontal safe pair at distance two. Otherwise the vertices u_4^W, u_4, u_4^E will be inside cop house R_2 and are therefore not part of the same safe set. Since every safe pair has a vertex inside the cop house, no pair is part of a safe set containing 4 vertices.

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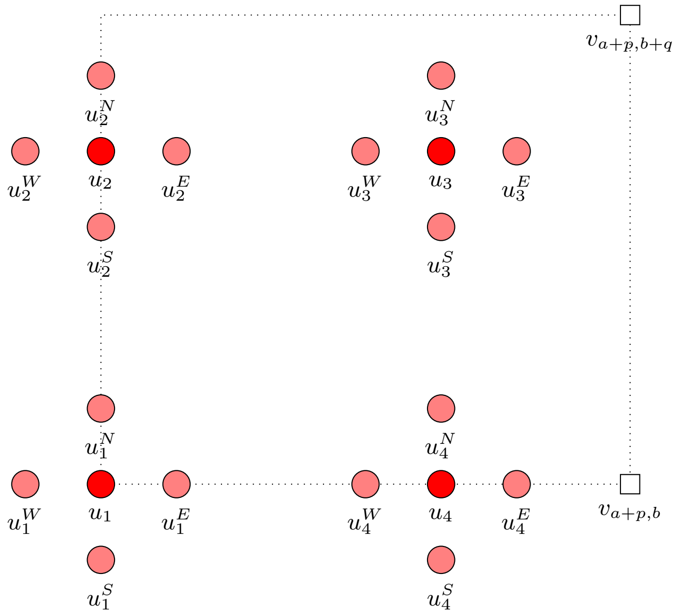

FIGURE 5.3: *The vertices in $N[O]$ where the two probed vertices are squares and the cop house R_2 is indicated with a dotted square. Note that $v_{a,b} = u_1, v_{a,b+d_j} = u_2, v_{a+d_i,b+d_j} = u_3$ and $v_{a+d_i,b} = u_4$.*

If $d_i \leq p-2$, then u_2^W is the only neighbour of u_2 and u_3 outside of R_2. Otherwise a similar argument can be applied to the vertices in $N[\{u_2, u_3\}]$. Therefore all safe sets in $N[O_1]$ contain two vertices in the same row or column, a distance of one or two apart.

In the next turn, the Cop chooses B_3 such that the Robber is localized to a diagonal safe pair. This probe will depend on the robber set O_2, where $O_2 = \{v_{x,y}, v_{x+1,y}\}$, $O_2 = \{v_{x,y}, v_{x+2,y}\}$ or $O_2 = \{v_{x,y}, v_{x,y-2}\}$.

Case 5.9.1 $O_2 = \{v_{x,y}, v_{x+1,y}\}$. The Cop probes $B_3 = \{v_{x-1,y+1}, v_{x,y}\}$ such that the distances from vertices in $N[O_2]$ are given in Table 5.1. It can be seen that the only safe sets, are diagonal safe pairs.

TABLE 5.1: *The distances from vertices in $N[O_2]$ to B_3 for Case 5.9.1. Note that the only safe sets, are diagonal safe pairs.*

$v \in N[O_2]$	$v_{x,y}$	$v_{x-1,y}$	$v_{x+1,y}$	$v_{x,y-1}$	$v_{x,y+1}$	$v_{x+1,y+1}$	$v_{x+1,y-1}$	$v_{x+2,y}$
$\vec{D}(B_3, v)$, $p = 1$	$[2,0]$	$[1,1]$	$[2,1]$	$[3,1]$	$[1,1]$	$[1,2]$	$[3,2]$	N/A
$\vec{D}(B_3, v)$, $p = 2$	$[2,0]$	$[1,1]$	$[3,1]$	$[3,1]$	$[1,1]$	$[2,2]$	$[4,2]$	$[3,2]$
$\vec{D}(B_3, v)$, $p \geq 3$	$[2,0]$	$[1,1]$	$[3,1]$	$[3,1]$	$[1,1]$	$[2,2]$	$[4,2]$	$[4,2]$

Case 5.9.2 $O_2 = \{v_{x,y}, v_{x+2,y}\}$. Note that this case only holds for $p \geq 2$. The Cop probes $B_3 = \{v_{x+1,y+1}, v_{x,y}\}$, where the distances from vertices in $N[O_2]$ to B_3 are given in Figure 5.4. It can again be seen that the only safe sets, are diagonal safe pairs. Note that if $p = 2$, then $\vec{D}(B_3, v_{x+2,y}) = [3,2]$ and not $[3,3]$.

Case 5.9.3 $O_2 = \{v_{x,y}, v_{x,y-2}\}$. The Cop now probes $B_3 = \{v_{x,y}, v_{x-1,y+1}\}$. The distances from vertices in $N[O_2]$ to B_3 are given in Table 5.2 for $p \geq 2$. These distances are given in Table 5.3 for $p = 1$. In both tables it can be seen that the only safe sets are diagonal safe pairs.

Now say the Robber is localized to a set $O_3 = \{v_{x,y}, v_{x+1,y+1}\}$. If $p \geq 2$, the Cop probes

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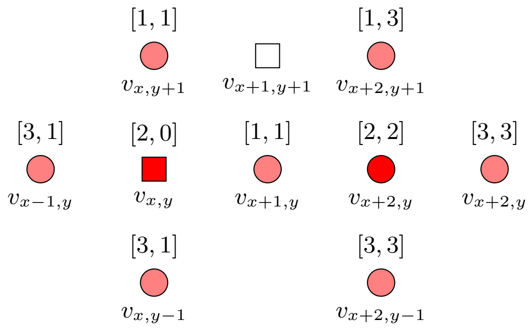

FIGURE 5.4: *An illustration of probe B_3 in Case 5.9.2. It can again be seen that the only safe sets, are diagonal safe pairs. Note that if $p = 2$, then $\vec{D}(B_3, v_{x+2,y}) = [3, 2]$ and not $[3, 3]$.*

TABLE 5.2: *The distances from vertices in $N[O_2]$ to B_3 for Case 5.9.3 when $p \geq 2$. Note that the only safe sets, are diagonal safe pairs.*

$v \in N[O_2]$	$v_{x,y}$	$v_{x-1,y}$	$v_{x+1,y}$	$v_{x,y+1}$	$v_{x,y-1}$	$v_{x,y-2}$	$v_{x-1,y-2}$	$v_{x+1,y-2}$	$v_{x,y-3}$
$\vec{D}(B_3, v)$, $q = 2$	[0, 2]	[1, 1]	[1, 3]	[1, 1]	[1, 3]	[2, 2]	[3, 1]	[3, 3]	N/A
$\vec{D}(B_3, v)$, $q = 3$	[0, 2]	[1, 1]	[1, 3]	[1, 1]	[1, 3]	[2, 4]	[3, 3]	[3, 5]	[3, 3]

TABLE 5.3: *The distances from vertices in $N[O_2]$ to B_3 for Case 5.9.3 when $p = 1$. Note that the only safe sets, are diagonal safe pairs.*

$v \in N[O_2]$	$v_{x,y}$	$v_{x-1,y}$	$v_{x+1,y}$	$v_{x,y+1}$	$v_{x,y-1}$	$v_{x,y-2}$	$v_{x-1,y-2}$	$v_{x+1,y-2}$	$v_{x,y-3}$
$\vec{D}(B_3, v)$, $q = 2$	[0, 2]	[1, 1]	[1, 2]	[1, 1]	[1, 3]	[2, 2]	[3, 1]	[3, 2]	N/A
$\vec{D}(B_3, v)$, $q = 3$	[0, 2]	[1, 1]	[1, 2]	[1, 1]	[1, 3]	[2, 4]	[3, 3]	[3, 4]	[3, 3]

$B_4 = \{v_{x-p+1,y}, v_{x-p,y-1}\}$ such that the distances from vertices in $N[O_3]$ to B_4 are given in Figure 5.5.

If $p = 1$, the Cop probes $B_4 = \{v_{x-1,y}, v_{x,y-1}\}$ such that the distances from vertices in $N[O_3]$ to B_4 are given in Table 5.4. Note that if $q = 2$, then $\vec{D}(B_4, v_{x+1,y+2}) = [2, 4]$ at not $[3, 4]$.

TABLE 5.4: *The distances from vertices in $N[O_3]$ to B_4 for $p = 1$ as in the proof of Proposition 5.9. Note that if $q = 2$, then $\vec{D}(B_4, v_{x+1,y+2}) = [2, 4]$ at not $[3, 4]$.*

$v \in N[O_3]$	$v_{x,y}$	$v_{x+1,y+1}$	$v_{x,y-1}$	$v_{x-1,y}$	$v_{x+1,y}$	$v_{x-1,y+1}$	$v_{x,y+1}$	$v_{x+1,y+2}$
$\vec{D}(B_4, v)$	[1, 1]	[2, 3]	[2, 0]	[0, 2]	[1, 2]	[1, 3]	[2, 2]	[3, 4]

All vertices in $N[O_3]$ are uniquely defined by their distance to B_4 and hence the Cop wins. If the Robber was localized to $O_3 = \{v_{x,y}, v_{x+1,y-1}\}$, the Cop probes $B_4 = \{v_{x-p+1,y}, v_{x-p,y+1}\}$ if $p \geq 2$ and $B_4 = \{v_{x-1,y}, v_{x,y+1}\}$ if $p = 1$ such that results follow similarly. $\square$

5.3 Even by even

For even m and n, first consider the case where $m \geq n \geq 8$:

Proposition 5.10. *Let $C_{2p} \square C_{2q}$ be the product of cycles with $p, q \geq 4$ and $p \geq q$. Then $\zeta(C_{2p} \square C_{2q}) = 2$.*

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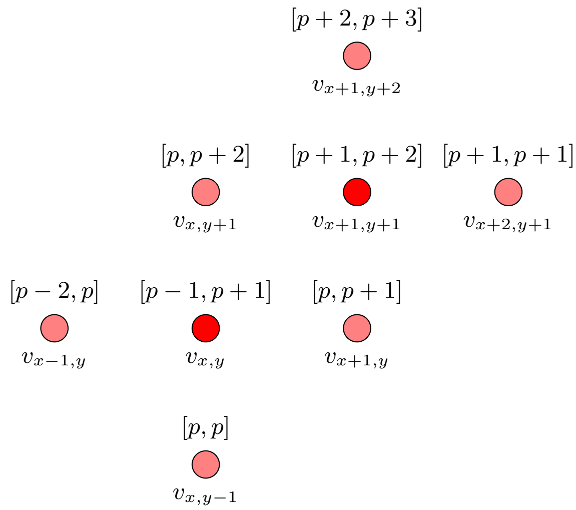

FIGURE 5.5: *The distances from vertices in $N[O_3]$ to B_4 for $p \geq 2$ as in the proof of Proposition 5.9. It can be seen that all vertices in $N[O_3]$ are resolved by B_4 and hence the Cop wins.*

Proof. In the first turn, the Cop probes $B_1 = \{v_{p,2q-1}, v_{p,q-1}\}$ such that the Robber is localized to $O_1 = \{v_{i,j}, v_{2p-i,j}, v_{i,2q-2-j}, v_{2p-i,2q-2-j}\}$ for $i \in \{0, 1, \ldots, 2p\}$ and $j \in \{0, 1, \ldots, 2q-1\}$ by Corollary 5.8. The Cop's second probe depends on d_i and d_j, where $d_i = d(v_{i,j}, v_{2p-i,j})$ and $d_j = d(v_{i,j}, v_{i,2q-2-j})$. These two distances are calculated as follows: $d_j = \min\{2j + 2, 2q - 2 - 2j\}$ as before and $d_i = \min\{2i, 2p - 2i\}$. The Robber set is again given by $O_1 = \{v_{a,b}, v_{a+d_i,b}, v_{a,b+d_j}, v_{a+d_i,b+d_j}\}$.

Strategy 5.10.1 $\mathbf{d_i \leq p - 2}$ and $\mathbf{d_j \leq q - 2}$. The Cop probes $B_2 = \{v_{a-1+p,b-1+q}, v_{a-1+p,b-1}\}$ such that B_2 is a translation of B_1. Let f be a translation such that $B_2 = f(B_1)$ and let R_2 be the set of all vertices $v_{w,z}$ where $w = a-1, a, \ldots, a-1+p$ and $z = b-1, b, \ldots, b-1+q$. Then $R_2 = f(R_1)$ such that it is a cop house by Corollary 5.8. Since $a + d_i \leq a + p - 2$ and $b + d_j \leq b + q - 2$, the neighbourhood $N[O_1]$ is contained in R_2 and therefore the Cop wins.

Strategy 5.10.2 $\mathbf{d_i > p - 2}$ or $\mathbf{d_j > q - 2}$. This means that at least one of the following holds: $d_i \in \{p-1, p\}$ or $d_j \in \{q-1, q\}$. Note that in the proof of Proposition 5.9 we have that $d_j > q-2$ and therefore a similar strategy can be used here. The Cop now probes $B'_2 = \{v_{a+p,b+q}, v_{a+p,b}\}$ as in Equation (5.6) such that we again have that each safe set is either a horizontal or vertical safe pair.

Notice that two vertices v_{i_1,j_1}, v_{i_2,j_2} are in the same safe set if and only if $i_1 + i_2 \equiv 2x \bmod (2p)$. Therefore the same argument as in the proof of Proposition 5.9 can be used to show that if vertices v_{i_1,j_1} and v_{i_2,j_2} are not part of the same neighbourhood $N[u_i]$, they are not part of the same safe set. Since every safe set is a vertical or diagonal pair of distance one or two, the Cop wins in the next turn by using Strategy 5.10.1. □

Now the consider the case where $n = 6$:

Proposition 5.11. *Let $C_{2p} \square C_6$ be a product of cycles with $p \geq 3$. Then $\zeta(C_{2p} \square C_6) = 2$.*

Proof. The Cop plays with two cops by using the imagined localization game on $C_{2p+1} \square C_6$. In the first turn, the Cop probes $B_1 = \{v_{p,5}, v_{p,2}\}$ as in the imagined game. Similarly as in the proof

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of Lemma 5.6, it can be shown that all safe sets have the form $O_1 = \{v_{i,j}, v_{2p-i,j}, v_{i,4-j}, v_{2p-i,4-j}\}$. This can equivalently be written as $O_1 = \{v_{i,j}, v_{i+d_i,j}, v_{i,j+d_j}, v_{i+d_i,j+d_j}\}$ where $d_i = d(v_{i,j}, v_{2p-i,j})$ and $d_j = d(v_{i,j}, v_{i,4-j})$ for $i \in \{0, 1, \dots, 2p-1\}$ and $j \in \{0, 1, \dots, 5\}$. Therefore the safe sets in the real game are the same as in the imagined game with the exception that $i \in \{0, 1, \dots, 2p\}$ in the imagined game. Therefore all robber sets in the real game are possible in the imagined game. For the second turn in the imagined game the Cop uses the strategy used in the second probe of the proof of Proposition 5.9 to localize the Robber to a safe set containing only two vertices, a distance of one or two apart. Since $i \leq 2p$, the real game does not contain horizontal safe pairs at distance 1 and hence the Robber is localized to a robber set of the form $O_2 = \{v_{x,y}, v_{x+2,y}\}$ or $O_2 = \{v_{x,y}, v_{x,y-2}\}$. These two cases are possible in the imagined game and handled in Cases 5.9.2 and 5.9.3 for the Cop's next probe. Thus the Robber is localized to a diagonal safe pair O_3.

If $O_3 = \{v_{x,y}, v_{x-1,y-1}\}$ the imagination strategy is not used and the Cop instead probes $B_4 = \{v_{x+1,y+1}, v_{x+1,y-2}\}$. The explicit distances from the vertices in $N[O_3]$ to B_4 are given in Figure 5.6, where it can be seen that no two distances are the same. The index of a vertex is shown below the vertex and its distance to B_4 is shown above it. The vertices of B_4 are squares, the vertices in O_3 are darker red and the vertices in $N(O_3)$ in lighter red.

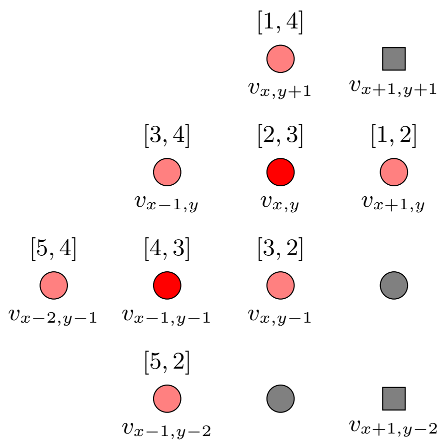

FIGURE 5.6: *The neighbourhood $N[O_3]$ and probe B_4 as in the proof of Proposition 5.11. The index of a vertex is shown below the vertex and its distance to B_4 is shown above it. The vertices of B_4 are squares, the vertices in O_3 are darker red and the vertices in $N(O_3)$ in lighter red. Note that no two distances are the same.*

Note that if $O_3 = \{v_{x,y}, v_{x+1,y-1}\}$, the Cop probes $B_4 = \{v_{x-1,y+1}, v_{x-1,y-2}\}$ and results follow similarly. Thus the Robber is located and the Cop wins. $\square$

In order to calculate the localization number of $C_{2p} \square C_4$, the following lemmas are used:

Lemma 5.12 [3]. *Let G be a bipartite graph, where $v \in V(G)$ and $w \in N(v)$. Say the Cop probes $B = \{b_1, b_2, \dots, \}$ in some turn and let $d_i = d(b_i, v)$. Then $d(b_i, w) \in \{d_i - 1, d_i + 1\}$.*

Lemma 5.13 [12]. *Let $C_m \square C_n$ be the product of cycles where $m, n \geq 3$. Then*

$$\dim(C_m \square C_n) = \begin{cases} 3 & \text{if } m \text{ or } n \text{ is odd} \\ 4 & \text{otherwise.} \end{cases}$$

Proposition 5.14. *Let $C_{2p} \square C_4$ be a product of cycles with $p \geq 2$. Then $\zeta(C_{2p} \square C_4) > 2$.*

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Proof. Assume that the Cop probes $B = \{b_1, b_2\}$. Then there are only three types of probes:

Type 1: The projection of B onto C_4 is a single vertex.

Type 2: In the projection of B onto C_4, the vertices of the projection are adjacent.

Type 3: In the projection of B onto C_4, the vertices of the projection are distance two apart.

It follows from Corollary 3.23 that for probes of Type 1 and 3 that every column will contain a safe pair and from the structure of C_4 it follows that this will be a vertical safe pair S_2^v. Also from Corollary 3.23 for a probe of Type 2 every column is resolved by the probe. Therefore, since $\dim(C_{2p}\square C_4) > 2$ by Lemma 5.13 there must exist a safe pair. We will show that for a probe of Type 2, every two adjacent columns contain two diagonal safe pairs.

Let B be of Type 2. Without loss of generality, assume that $b_1 = v_{i,3}$ and $b_2 = v_{i,2}$ for some column i. Now consider a column k and say $\vec{D}(B, v_{k,3}) = [d_1, d_2]$. Then by Lemma 5.12 and the structure of C_4, the distances from B to the vertices in column k are given in Table 5.5.

TABLE 5.5: *The distances from $B = \{v_{i,3}, v_{i,2}\}$ to the vertices in column k for $C_{2p}\square C_4$.*

v	$\vec{D}(B, v)$
$v_{k,3}$	$[d_1, d_2]$
$v_{k,2}$	$[d_1+1, d_2-1]$
$v_{k,1}$	$[d_1+2, d_2]$
$v_{k,0}$	$[d_1+1, d_2+1]$

Now compare this to the distances from B to the vertices in column $k+1$ as given in Table 5.6. The table gives all the possible distances to the vertices in column $k+1$, as it follows from Lemma 5.12.

TABLE 5.6: *All possible distances from $B = \{v_{i,3}, v_{i,2}\}$ to the vertices in column $k+1$ for $C_{2p}\square C_4$ as by Lemma 5.12.*

v	$\vec{D}(B, v)$	$\vec{D}(B, v)$	$\vec{D}(B, v)$	$\vec{D}(B, v)$
$v_{k+1,3}$	$[d_1+1, d_2+1]$	$[d_1+1, d_2-1]$	$[d_1-1, d_2+1]$	$[d_1-1, d_2-1]$
$v_{k+1,2}$	$[d_1+2, d_2]$	$[d_1+2, d_2-2]$	$[d_1, d_2]$	$[d_1, d_2-2]$
$v_{k+1,1}$	$[d_1+3, d_2+1]$	$[d_1+3, d_2-1]$	$[d_1+1, d_2+1]$	$[d_1+1, d_2-1]$
$v_{k+1,0}$	$[d_1+2, d_2+2]$	$[d_1+2, d_2]$	$[d_1, d_2+2]$	$[d_1, d_2]$

It is clear from the tables that every two adjacent columns contain two diagonal safe pairs. We now consider two possibilities of a Robber set for the second turn:

Strategy 5.14.1 Diagonal safe pair. If probe B_2 is of Type 1 or 3, a vertical safe pair will exist. This safe pair will either contain a vertex in row one and row three, or contain a vertex in row two and row four. It follows that the Robber can move to a safe pair in the next round. If probe B_2 is of Type 2, there is at least one other diagonal safe pair to move to.

Strategy 5.14.2 Vertical safe pair. As in the previous strategy, if B_2 is of Type 1 or 3 the Robber will either be safe or be able to move to a vertical safe pair. Hence assume that $B_2 = \{a_1, a_2\}$ is of Type 2. If a_1 is in the same row as the Robber, then a diagonal safe pair will exist in columns a_1 and $a_1 + 1$ by Tables 5.5 and 5.6. Otherwise if a_1 is not in the same row as the Robber, it again follows from Tables 5.5 and 5.6 that the Robber will be able to move to a diagonal safe pair.

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The Robber can therefore perpetually avoid capture by using Strategies 5.14.1 and 5.14.2. $\square$

Corollary 5.15. *Let P_m be the path of order $m \geq 2$ and C_4 be the cycle of order 4. Then $\zeta(P_m \square C_4) > 2$.*

Proof. Note that all paths are bipartite graphs and therefore $P_m \square C_4$ is a bipartite graph by Lemma 3.16. The proof of Proposition 5.14 can then directly be applied to $P_m \square C_4$, since the cyclic nature of the rows of $C_{2p} \square C_4$ was never used in the proof. $\square$

Proposition 5.16. *Let $C_{2p} \square C_4$ be the product of cycles with $p \geq 2$. Then $\zeta(C_{2p} \square C_4) \leq 3$.*

Proof. This already holds for $p \geq 4$ by Equation (5.1). Let $p = 3$ and say the Cop probes $B_1 = \{v_{0,3}, v_{0,1}, v_{1,3}\}$ such that the distances to the vertices are given in Table 5.7.

TABLE 5.7: *The distances $\vec{D}(B_1, v_{i,j})$ from probe B_1 to vertices $v_{i,j}$ in $C_6 \square C_4$.*

$j = 3$	$[0, 2, 1]$	$[1, 3, 0]$	$[2, 4, 1]$	$[3, 5, 2]$	$[2, 4, 3]$	$[1, 3, 2]$
$j = 2$	$[1, 1, 2]$	$[2, 2, 1]$	$[3, 3, 2]$	$[4, 4, 3]$	$[3, 3, 4]$	$[2, 2, 3]$
$j = 1$	$[2, 0, 3]$	$[3, 1, 2]$	$[4, 2, 3]$	$[5, 3, 4]$	$[4, 2, 5]$	$[3, 1, 4]$
$j = 0$	$[1, 1, 2]$	$[2, 2, 1]$	$[3, 3, 2]$	$[4, 4, 3]$	$[3, 3, 4]$	$[2, 2, 3]$
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$

From the table it can be seen that all safe sets have the form $\{v_{i,0}, v_{i,2}\}$ for $i \in \{0, 1, \dots, 5\}$. In the second turn, the Cop probes $B_2 = \{v_{i,0}, v_{i-1,1}, v_{i+1,1}\}$ such that $N[O_1]$ is resolved as shown in Table 5.8.

TABLE 5.8: *The distances from the vertices in $N[O_1]$ to probe B_2 for $C_6 \square C_4$.*

$v_{i,j} \in N[O_1]$	$v_{i,0}$	$v_{i-1,0}$	$v_{i+1,0}$	$v_{i,1}$	$v_{i,2}$	$v_{i-1,2}$	$v_{i+1,2}$	$v_{i,3}$
$\vec{D}(B_2, v_{i,j})$	$[0, 2, 2]$	$[1, 1, 2]$	$[1, 3, 1]$	$[1, 1, 1]$	$[2, 2, 2]$	$[3, 1, 3]$	$[3, 3, 1]$	$[1, 3, 3]$

The case when $p = 2$ follows in a similar fashion such that three cops are enough for $p \geq 2$. $\square$

Propositions 5.14 and 5.16 together prove that $\zeta(C_{2p} \square C_4) = 3$. This completes all cases for m and n such that Theorem 5.1 has been proved.

5.4 General products with C_n

Next, consider the Cartesian product $G_m \square C_n$ where G_m is a connected graph of order m and C_n the cycle of order $n \geq 3$. The cases when $m = 1$ and $G_m = C_m$ are covered in Theorems 3.13 and 5.1. Next, say $G_m = P_m$. Cáceres et al. [12] proved that $\psi(P_m) = 2$ and therefore $\zeta(P_m \square C_n) = 2$ for $n \geq 7$ by Proposition 3.19 and Corollary 3.25. The localization number of $P_m \square C_n$ for $n \leq 6$ remains to be calculated. For odd n we have $\zeta(P_m \square C_n) \leq 2$ such that $\zeta(P_m \square C_n) = 2$ by Lemma 5.2. It follows that

$$\zeta(P_m \square C_n) = 2 \tag{5.7}$$

for $n \geq 7$ and odd n. Thus the only cases to be considered for the localization number of $P_m \square C_n$ is $n \in \{4, 6\}$.

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Proposition 5.17. *Let P_m be the path of order $m \geq 2$ and C_4 be the cycle of order four. Then $\zeta(P_m \square C_4) = 3$.*

Proof. By Corollary 5.15 it only needs to be shown that three cops are enough for the Cop to guarantee a win. Cáceres et al. [12] showed that $\dim(H) \leq \dim(P_m \square H) \leq \dim(H) + 1$. Since $\dim(C_n) = 2$, we have that $\dim(P_m \square C_4) \leq 3$ and therefore the Cop can win the localization game in one turn on $P_m \square C_4$ using three cops. $\square$

Proposition 5.18. *Let P_m be the path of order $m \geq 2$ and C_6 be a cycle of order six. Then $\zeta(P_m \square C_6) = 2$.*

Proof. By Proposition 3.19, we only need to show that two cops are enough to guarantee a win. In the first turn, the Cop probes $B_1 = \{v_{0,0}, v_{0,5}\}$. Let $v_{i,j}$ be any vertex in $P_m \square C_6$. The distance to B_1 is given by

$$\vec{D}(B_1, v_{i,j}) = \begin{cases} [i+j, i+j+1] & \text{for } j = 0,1,2 \\ [6+i-j, 5+i-j] & \text{for } j = 3,4,5. \end{cases} \tag{5.8}$$

By Equation (5.8), it is clear that there are two safe houses: SH_1 in rows $j = 0,1,2$ and SH_{-1} in rows $j = 3,4,5$. Also two vertices v_{i_1,j_1} and v_{i_2,j_2} in safe house SH_1 are part of the same safe set if $i_1 - j_1 = i_2 - j_2$ such that SH_1 has negative diagonal safe sets. Similarly two vertices in SH_{-1} are part of the same safe set if $i_1 + j_1 = i_2 + j_2$ such that safe house SH_{-1} has positive diagonal safe sets. The probe B_1, distances to B_1, safe sets and safe houses are illustrated in Figure 5.7 for $P_6 \square C_6$.

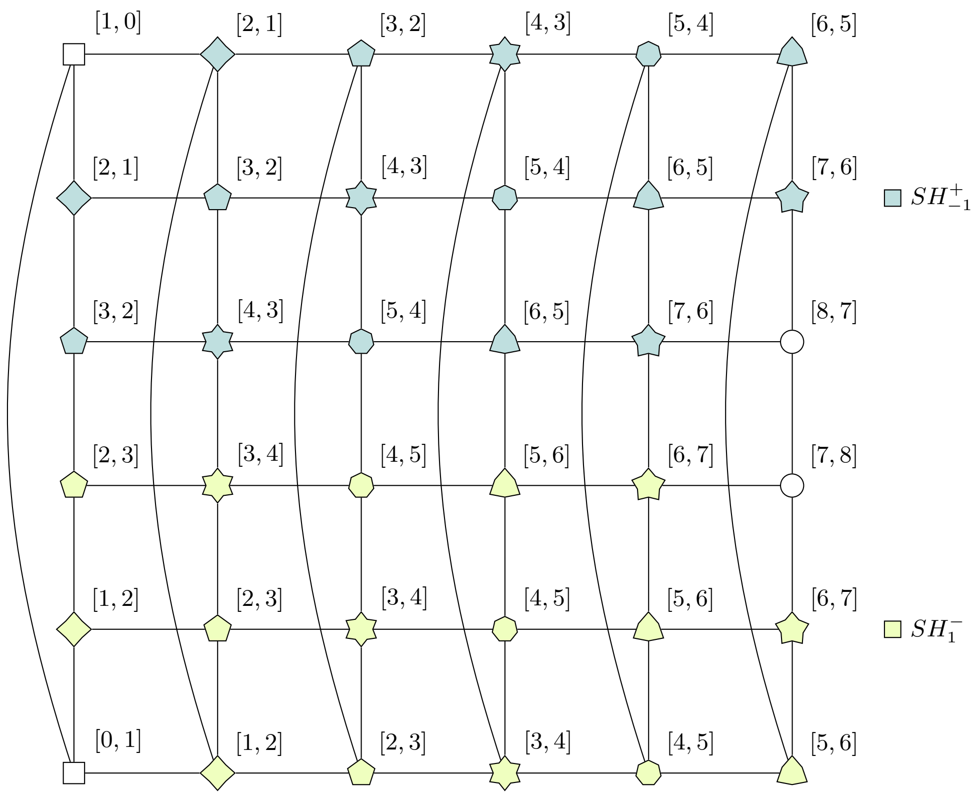

FIGURE 5.7: *The graph $P_6 \square C_6$ with probe B_1 indicated as squares as in the proof of Proposition 5.18. Vertices with the same colour and shape are in the same safe set and empty vertices are not part of a safe set. The distance from a vertex to B_1 is indicated above the vertex. Safe houses SH^-_1 and SH^+_{-1} are also indicated.*

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If the Robber is at vertices $v_{0,0}$, $v_{0,5}$, $v_{m,2}$ or $v_{m,3}$, the Cop wins immediately. If not, the Cop probes $B_2 = \{v_{0,3}, v_{0,2}\}$ such that B_2 is equal to B_1 translated with three vertices downwards. This also means that safe house $SH_1^-(B_2)$ lies in rows $3, 4, 5$ and that safe house $SH_{-1}^+(B_2)$ lies in rows $j = 0, 1, 2$. Say the Robber is localized to robber set $O_1 = \{v_{i,j}, v_{i+1,j+1}, \ldots, v_{i+k-1,j+k-1}\}$ in safe house SH_{-1}^+, with $k \in \{2, 3\}$. If vertices in $N[O_1]$ that lie in safe house $SH_1^-(B_2)$ are part of a safe set, it will be a negative diagonal safe pair since safe house $SH_1^-(B_2)$ only contains negative diagonal safe sets. By the choice of B_2, note that $O \subset SH_1^-(B_2)$ and that only vertices $v_{i,j-1}$ and $v_{i+k-1,j+k}$ in $N(O_1)$ can fall outside $SH_1^-(B_2)$. These vertices only fall outside safe house $SH_1^-(B_2)$ if $j = 3$ and $j+k-1 = 5$ respectively. Thus say $j = k = 3$ such that $v_{i,j-1} = v_{i,2}$ and $v_{i+k-1,j+k} = v_{i+2,0}$. Now since vertices $v_{i,2}$ and $v_{i+2,0}$ lie on the same negative diagonal and are in safe house SH_{-1}^+, they are not part of the same safe set. In a similar fashion it can be shown that if the Robber is localized to a robber set O in safe house SH_1^-, the only safe sets in $N[O_1]$ are positive diagonal safe pairs.

Therefore the only safe sets in $N[O_1]$ after probe B_2, are diagonal safe pairs. If the Robber is not uniquely located by probe B_2, the Cop requires a third probe. Say the Robber is localized to robber set $O_2 = \{v_{x,y}, v_{x-1,y-1}\}$ after probe B_2. The Cop can now probe $B_3 = \{v_{x+1,y+1}, v_{x+1,y-2}\}$ such that the distances from vertices in $N[O_2]$ to B_3 are given in Figure 5.8.

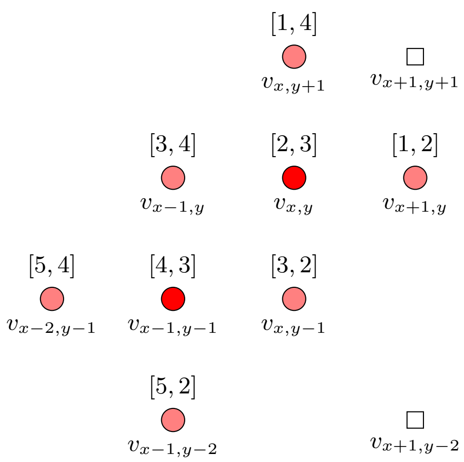

FIGURE 5.8: *Distances from vertices in $N[O_2]$ to probe B_3 as in the proof of Proposition 5.18.*

In the figure it can be seen that every distance is unique and thus the Cop wins. If $O = \{v_{x,y}, v_{x+1,y-1}\}$, then the Cop probes $B_3 = \{v_{x-1,y+1}, v_{x-1,y-2}\}$ and results follow similarly. Therefore the Cop can win on $P_m \square C_6$ using two cops in at most three turns if $m \geq 2$. $\square$

The following result follows from Equation (5.7) and Propositions 5.17 and 5.18:

Theorem 5.19. *Let P_m be the path of order $m \geq 2$ and C_n be a cycle of order $n \geq 3$. If $n = 4$, then $\zeta(P_m \square C_n) = 3$. If not, then $\zeta(P_m \square C_n) = 2$.*

Next we consider the case when $G_m = K_m$. If $m = 1$, then $G_m \square C_n = C_n$ such that Theorem 3.13 applies. If $m = 2$, then $G_m = P_m$ such that Theorem 5.19 applies. If $m = 3$, then $G_m = C_n$ such that Theorem 5.1 applies and if $m \geq 4$, then $\zeta(G_m \square C_n) = m - 1$ by Theorem 4.11. Thus $\zeta(K_m \square C_n)$ is known for all $m \geq 1$.

Lastly consider $G_m \square C_n$ where G_m is any non-complete connected graph of order m. By Theorem 3.13, $\zeta(C_n) = 1$ for $n \geq 7$ such that $\zeta(G_m \square C_n) \leq \psi(G_m) \leq m - 1$ by Corollary 3.25. Similarly,

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$\zeta(G_m \square C_n) \leq \psi(G_m) + 1 \leq m$ for $n \leq 6$. Since $G_m \neq K_m$, $\zeta(G_m) \leq m-2$ it follows from Theorem 3.24 and Lemma 5.2 that $\zeta(G_m \square C_n) \leq m-1$ for odd n. Note that if $G_m = K_m$, then $\zeta(G_m \square C_n \leq m-1$ by Theorem 4.11. Further, if $\operatorname{diam}(G_m) \neq 2$, then $\psi(G_m) \leq m-2$ by Proposition 3.29 such that $\zeta(G_m \square C_n) \leq m-1$. This proves the following proposition:

Proposition 5.20. *Let G_m be a connected graph of order $m \geq 4$ and C_n be a cycle of order $n \geq 3$. Then $\zeta(G_m \square C_n) \leq m$ if $n \in \{4, 6\}$ and diam$(G_m) = 2$ and $\zeta(G_m \square C_n) \leq m-1$ otherwise.*

5.5 Chapter summary

This chapter was started off by stating the main result for the product of two cycles, namely that $\zeta(C_m \square C_n)$ is mostly equal to 2. We then proceed to prove this in three cases: odd by odd, odd by even and even by even.

In the last section, we consider $G_m \square C_n$. First we showed that $\zeta(P_m \square C_n)$ is equal to 2, except if $n = 4$. Lastly we provided and upper bound on $\zeta(G_m \square C_n)$.

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CHAPTER 6

Products with S_m

In this chapter we investigate the localization number of the product of two stars, where $S_m = K_{1,m-1}$ denotes the star of order m. In Section 6.1 we prove that $\zeta(S_m \square S_m) = \lceil \frac{m}{2} \rceil - 1$ for $m \geq 7$ after which we show that $\zeta(S_m \square S_m) = \lfloor \frac{m}{2} \rfloor$ for $m \in \{4, 5, 6\}$.

The vertex set of S_m will be denoted as $V(S_m) = \{s_0, s_1, \ldots, s_{m-1}\}$ where s_0 is the universal vertex. Note that the graph S_m has two orbits: $\{s_0\}$ and $\{s_i \mid i \geq 1\}$. Thus, the product $S_m \square S_m$ has three orbits: $\{v_{0,0}\}$, $\{v_{a,0} \mid a \geq 1\} \cup \{v_{0,b} \mid b \geq 1\}$ and $\{v_{c,d} \mid c, d \geq 1\}$.

Recall that $\zeta(S_m) = 1$ and $\dim(S_m) = m - 1$ as by Equation (2.3). Thus the difference between $\zeta(G)$ and $\dim(G)$ for star graphs can be arbitrarily large. Further, as shown in Proposition 3.28, $\psi(S_m) = m - 1$. For complete graphs, cycles and paths the following statements hold:

$$\psi(G) - \zeta(G) \leq 1, \tag{6.1}$$

$$\zeta(G) \leq \zeta(G \square G) - 1 \text{ and} \tag{6.2}$$

$$\zeta(G) \leq \zeta(H) \implies \zeta(G \square G) \leq \zeta(H \square H). \tag{6.3}$$

This is however not true for stars. By Equation (2.3), the difference between $\zeta(S_m)$ and $\psi(S_m)$ can be arbitrarily large such that Equation (6.1) does not hold. In this chapter we show that Equations (6.2) and (6.3) do not hold for all stars. In fact, we show that the difference between $\zeta(G)$ and $\zeta(G \square G)$ can be arbitrarily large.

6.1 Products of stars with large order

It is first shown that $\lceil \frac{m}{2} \rceil - 2$ cops are not enough to locate the Robber on $S_m \square S_m$.

Proposition 6.1. *For the star S_m of order $m \geq 7$, $\zeta(S_m \square S_m) \geq \lceil \frac{m}{2} \rceil - 1$.*

Proof. For $a = \lceil \frac{m}{2} \rceil - 1$, we show that $a - 1$ cops are not enough to guarantee a win for the Cop. It is easy to see that for any choice of $a - 1$ probed vertices on S_m, at least $m - 1 - (a - 1) = m - a$ leaves will belong to the same safe set. Note that $m - a = a + 2$ for m even and $m - a = a + 1$ for m odd such that at least $a + 1$ leaves will be part of the same safe set in S_m. Therefore by Corollary 3.23, each row and column of $S_m \square S_m$ contains at least $a + 1$ vertices in the same safe set for any given $(a - 1)$-vertex probe. Note that these $a + 1$ vertices will be in the same orbit. We show that there exists a strategy where the Robber can repeatedly return to a safe set of $a + 1$ vertices contained in a row. Therefore, after probe B_1, we assume without loss of generality the existence of the following safe set: $O_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,a+1}\}$. It follows that in the next turn, the Robber can be at any vertices in O_1 as well as the vertices in $N(O_1) =$

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$\{v_{0,1}, v_{0,2}, \ldots, v_{0,a+1}\} \cup \{v_{1,0}\}$. Again by Corollary 3.23, irrespective of the Cop's choice of B_2, there will be at least $a+1-(a-1)=2$ vertices part of the same safe set in column 0. Without the loss of generality, we thus assume that $\{v_{0,1}, v_{0,2}\} \subseteq O_2$ such that $N[O_2]$ contains all vertices in rows 1 and 2 as well as vertex $v_{0,0}$. By Corollary 3.23, each of these rows will again contain at least $a+1$ vertices in the same safe set, as desired. □

Say some vertex $b_{i,j}$ in column $i>0$ and row $j>0$ is probed by the Cop. Then the distance $d(v_{x,y}, b_{i,j})$ to any vertex $v_{x,y} \neq b_{i,j}$ is given by

$$d(v_{x,y}, b_{i,j}) = \begin{cases} 1 & \text{if } (x=0,\ y=j) \text{ or } (x=i,\ y=0), \\ 2 & \text{if } (x=y=0) \text{ or } (x \neq \{0,i\},\ y=j) \text{ or } (x=i,\ y\neq\{0,j\}), \\ 3 & \text{if } (x=0,\ y\neq\{0,j\}) \text{ or } (x\neq\{0,i\},\ y=0), \\ 4 & \text{if } (x\neq\{0,i\},\ y\neq\{0,j\}). \end{cases} \tag{6.4}$$

The next lemma shows that if the robber set can be reduced to at most $m-2$ vertices contained in a single row (or column), then the cop can win with $\lceil \frac{m}{2} \rceil - 1$ cops.

Lemma 6.2. *Let S_m be the star of order $m \geq 7$ and consider $S_m \Box S_m$. Say the Cop plays with $\lceil \frac{m}{2} \rceil - 1$ cops and the Robber is localized to a safe set in a single row or column, containing at most $m-2$ vertices of degree two. Then the Cop wins.*

Proof. Let $a = \lceil \frac{m}{2} \rceil - 1$. Say the Robber is localized to $O_{t-1} = \{v_{1,j}, v_{2,j}, \ldots, v_{m-2,j}\}$ in turn $t-1$ such that O_{t-1} contains $m-2$ vertices in row j. Note that results follow similarly if O_{t-1} contained vertices in a single column. In the next turn, the Robber can be at vertices in O_{t-1} as well as $N(O_{t-1}) = \{v_{0,j}\} \cup \{v_{1,0}, v_{2,0}, \ldots, v_{m-2,0}\}$. The Cop then probes $B_t = \{v_{m-2,j}, v_{m-3,j}, \ldots, v_{m-a-1,j}\}$, resulting in two safe sets in $N[O_{t-1}]$: $\{v_{1,j}, v_{2,j}, \ldots, v_{m-a-2,j}\}$ and $\{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\}$.

Say $O_t = \{v_{1,j}, v_{2,j}, \ldots, v_{m-a-2,j}\}$ such that $N(O_t) = \{v_{0,j}\} \cup \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\}$. If the Cop translates B_t to form B_{t+1} such that $B_{t+1} = \{v_{m-a-2,j}, v_{m-a-3,j}, \ldots, v_{m-2a-1,j}\}$, the Cop wins. Note that $m-a-2=a$ for even m and $m-a-2=a-1$ for odd m such that $O_t \subseteq B_{t+1}$.

Now say $O_t = \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\}$ such that $N[O_t]$ contains all vertices in columns 1 to $m-a-2$ as well as vertex $v_{0,0}$. The Cop now probes $B_{t+1} = \{v_{1,m-2}, v_{1,m-1}, v_{2,m-1}, \ldots, v_{a-1,m-1}\}$. After this probe, there are $m-a-2$ safe sets of the form $O_{t+1} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m-3}\}$ where $i \in \{1, 2, \ldots, m-a-2\}$. Hence O_{t+1} is confined to a single column of degree two vertices containing one less vertex then O_{t-1}.

Therefore, if the robber set O_{t-1} contains $c \leq m-2$ vertices in a single row it is reduced to a robber set O_t of $c-1$ vertices in a single column, and vice versa. Thus, in every second turn, the Cop probes $B_c = \{v_{1,m-1}, v_{2,m-1}, \ldots, v_{c-1,m-1}\} \cup \{v_{1,m-2}, v_{1,m-3}, \ldots, v_{1,m-(a-c+1)-1}\}$ if the robber is localized to c vertices contained in a single column, and $B_r = \{v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,c-1}\} \cup \{v_{m-2,1}, v_{m-3,1}, \ldots, v_{m-(a-c+1)-1,1}\}$ if the Robber is localized to c vertices contained in a single row. The Cop can thus keep applying this strategy until the Robber set only contains one vertex such that the Cop wins. □

Corollary 6.3. *Say the Cop plays with $a = \lceil \frac{m}{2} \rceil - 1$ cops and the Robber is localized to a safe set in row or column 0, containing at most a vertices only of degree three. Then the Cop wins.*

Proof. Say the Robber is localized to robber set $O = \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\}$. If the Cop probes $B = \{v_{1,m-2}, v_{1,m-1}, v_{2,m-1}, \ldots, v_{a-1}\}$, the resulting safe sets will all be contained in a single row (or column) with at most $m-2$ vertices. The Cop then wins according to Lemma 6.2. □

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Lemma 6.4. *Let S_m be the star of order $m \geq 7$ and consider $S_m \square S_m$. Say the Cop plays with $\lceil \frac{m}{2} \rceil - 1$ cops and the Robber is localized to a safe set of the form $O = \{v_{d,0}, v_{0,d}\}$ where $d \geq 1$. Then the Cop wins.*

Proof. In the next turn, the Robber can be at any vertex in row and column d as well as vertex $v_{0,0}$. If $d > 1$, the Cop now probes $B = \{v_{d-1,d}, v_{d-1,d-1}, \ldots, v_{d-1,d-a-1}\}$ where $a = \lceil \frac{m}{2} \rceil - 1$ such that two safe sets exist. The first has distance vector $\vec{D}(B, r) = [2, 4, 4, \ldots, 4]$ and defines robber set $O' = \{v_{1,d}, v_{2,d}, \ldots, v_{m-1,d}\} \setminus \{v_{d-1,d}\}$. The second has distance vector $\vec{D}(B, r) = [4, 4, \ldots, 4]$ and defines robber set $O'' = \{v_{d,1}, v_{d,2}, \ldots, v_{d,d-a-1}\} \cup \{v_{d,d+1}, v_{d,d+2}, \ldots, v_{d,m-1}\}$. The Cop wins for O' and O'' by Lemma 6.2.

Note that if $d = 1$, then $B = \{v_{2,1}, v_{2,2}, \ldots, v_{2,a}\}$ such that results follow similarly. □

Lemma 6.5. *Say the Cop plays with $a = \lceil \frac{m}{2} \rceil - 1$ cops and the Robber is localized to safe pair $O = \{v_{f,g}, v_{g,f}\}$ where $f, g > 0$. Then the Cop wins.*

Proof. If $O = \{v_{f,g}, v_{g,f}\}$ where $f, g > 0$, then $N[O] = O \cup \{v_{f,0}, v_{g,0}, v_{0,f}, v_{0,g}\}$. In the next turn, the Cop probes $B = \{v_{g,f}, v_{g,g}, b_3, b_4, \ldots, b_a\}$ such that probes b_3 to b_a are vertices in column g, excluding vertex $v_{g,0}$. Now vertex $v_{0,f}$ is the only vertex in $N[O]$ adjacent to b_1 and $v_{0,g}$ is the only vertex in $N[O]$ adjacent to b_2. Further $v_{g,f} = b_1$, $v_{g,0}$ is the only vertex in $N[O]$ adjacent to all probes and lastly $v_{f,g}$ is the only vertex in $N[O]$ a distance of two from b_2. Thus $N[O]$ is resolved. □

Proposition 6.6. *For the star S_m or order $m \geq 7$, $\zeta(S_m \square S_m) \leq \lceil \frac{m}{2} \rceil - 1$.*

Proof. Say the Cop plays with $a = \lceil \frac{m}{2} \rceil - 1$ cops. In the first turn, the Cop probes $B_1 = \{v_{m-1,m-1}, v_{m-1,m-2}, \ldots, v_{m-1,m-a}\}$. By Equation (6.4) the vertices in column 0 are a distance of 3 from all probes, except the probe in their row. It follows that the vertices in column 0 with a probe in their row are resolved by B_1. Vertex $v_{m-1,0}$ is the only vertex to neighbour all probes and is therefore not part of a safe set. The remaining vertices can be divided into four classes of distance vectors, given in Table 6.1.

TABLE 6.1: *The 4 classes of safe sets after probe B_1*

Class	$\vec{D}(B_1, v_{x,y})$	Range
1	$[4, \ldots, 4, 2, 4, \ldots, 4]$	$1 \leq x \leq m-2$, $y \in \{m-a, m-a+1, \ldots, m-1\}$
2	$[4, 4, \ldots, 4]$	$1 \leq x \leq m-2$, $1 \leq y < m-a$
3	$[2, 2, \ldots, 2]$	$(x = y = 0)$ and $(x = m-1,\ 1 \leq y < m-a)$
4	$[3, 3, \ldots, 3]$	$(x = 0,\ 1 \leq y < m-a)$ and $(1 \leq x \leq m-2,\ y = 0)$

Note that for the first class, all distances in the vector are 4, except the y'th distance which is equal to 2. Probe B_1 and resulting distance vectors are illustrated for $m = 7$ in Figure 6.1.

The Cop wins by Lemma 6.2 if the Robber is localized to a safe set of the first class. If the Robber is localized to a safe set in the second class, the Cop uses the following strategy:

Strategy 6.6.1 $\vec{D}(B_1, v_{x,y}) = [4, 4, \ldots, 4]$**.** In this case, the Robber can be at any vertex in rows 0 to $m-a-1$, except $v_{0,0}$ and those in column $m-1$. The Cop now probes $B_2 = \{v_{m-2,m-a-1}, v_{m-3,m-a-1}, \ldots, v_{m-a-1,m-a-1}\}$. Since B_2 is a translation of B_1, the same four distance vectors in Table 6.1 exist here. If $\vec{D}(B_2, v_{x,y}) = [4, \ldots, 4, 2, 4, \ldots, 4]$, then $O_2 = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m-a-2}\}$ for $i \in \{m-2, m-3, \ldots, m-a-1\}$, and if $\vec{D}(B_2, v_{x,y}) = [2, 2, \ldots, 2]$, then $O_2 = \{v_{1,m-a-1}, v_{2,m-a-1}, \ldots, v_{m-a-2,m-a-1}\}$. In both cases the Cop wins by Lemma 6.2.

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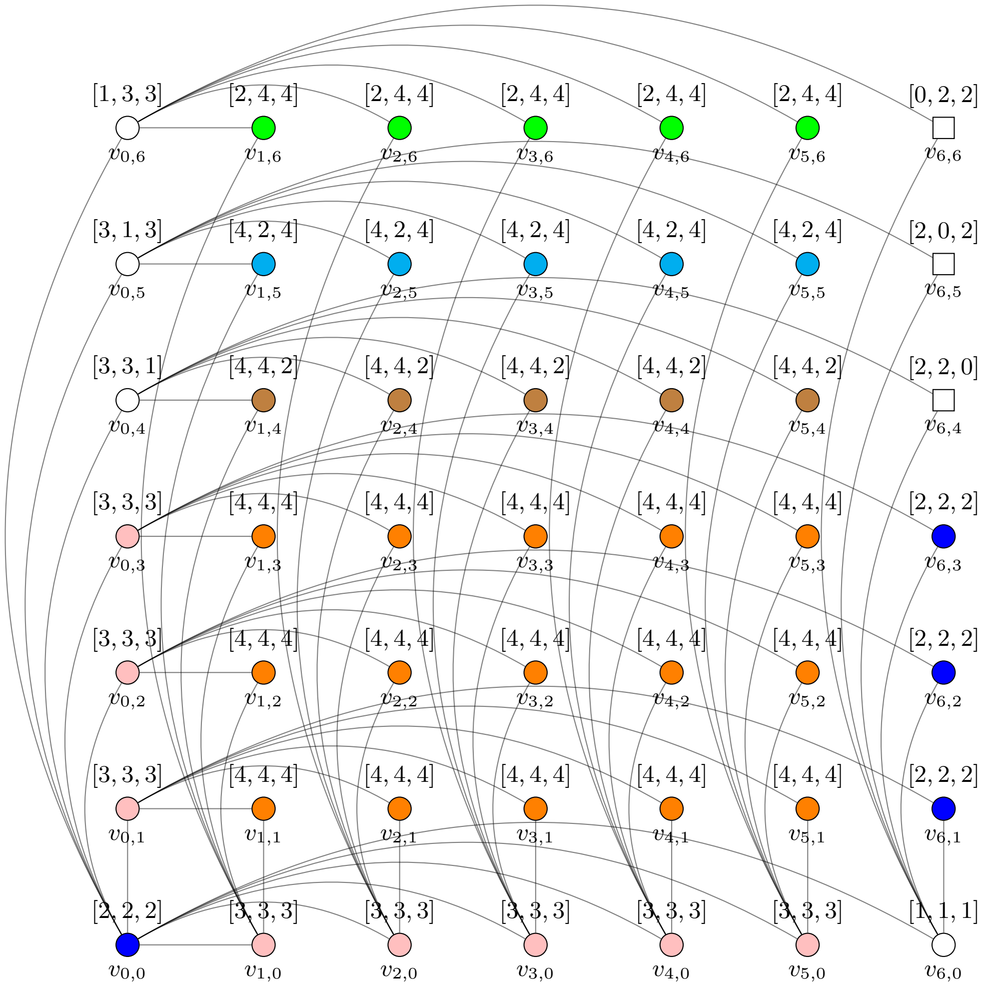

FIGURE 6.1: *The graph $S_7 \square S_7$ with probe B_1 as in Proposition 6.6.*

Say $\vec{D}(B_2, v_{x,y}) = [4, 4, \ldots, 4]$ such that $O_2 = \{v_{d,e} \mid 1 \leq d, e \leq m - a - 2\}$ and $N[O_2] = \{v_{d,e} \mid 0 \leq d, e \leq m - a - 2\} \setminus \{v_{0,0}\}$. Next, the Cop probes $B_3 = \{v_{1,m-a-2}, v_{2,m-a-2}, \ldots, v_{a,m-a-2}\}$. Since $m - a - 2 \leq a$, there are two classes of distance vectors in $N[O_2]$: $[4, \ldots, 4, 2, 4, \ldots, 4]$ and $[3, 3, \ldots, 3]$. In the first case, $O_3 = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m-a-3}\}$ such that the Cop wins by Lemma 6.2. In the second, $O_3 = \{v_{0,1}, v_{0,2}, \ldots, v_{0,m-a-3}\}$ and the Cop wins by Corollary 6.3.

Lastly, say $\vec{D}(B_2, v_{x,y}) = [3, 3, \ldots, 3]$, where it follows that $O_2 = \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\} \cup \{v_{0,1}, v_{0,2}, \ldots, v_{0,m-a-2}\}$. At the start of the third turn, the Robber can be at any vertices in rows and columns 1 to $m-a-2$ as well as vertex $v_{0,0}$. The Cop probes $B_3 = \{v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,a}\}$. Since $a \geq m - a - 2$, there are three classes of distance vectors in $N[O_2]$: $[4, \ldots, 4, 2, 4, \ldots, 4]$, $[4, 4, \ldots, 4]$ and $[3, 3, \ldots, 3]$. For the first class, the Robber is localized to robber set $O_3 = \{v_{1,j}, v_{2,j}, \ldots, v_{m-2,j}\}$ where $j \in \{1, 2, \ldots, m - a - 2\}$ such that the Cop wins by Lemma 6.2. Note that if m is odd, there is another smaller safe set of the same class in row a which can also by solved by Lemma 6.2. If $\vec{D}(B_3, v_{x,y}) = [3, 3, \ldots, 3]$, then $O_3 = \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-2,0}\}$ and the Cop wins by Corollary 6.3. The last possible distance vector is $\vec{D}(B_3, v_{x,y}) = [4, 4, \ldots, 4]$ such that $O_3 = \{v_{d,e} \mid 1 \leq d \leq m - a - 2,\ m - a - 1 \leq e \leq m - 1\}$. In this case, the Cop next probes $B_4 = \{v_{1,m-1}, v_{2,m-1}, \ldots, v_{a,m-1}\}$ such that he wins by Lemma 6.2 and Corollary 6.3.

Next, we consider the third type of distance vector after probe B_1:

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Strategy 6.6.2 $\vec{D}(B_1, v_{x,y}) = [2, 2, \ldots, 2]$**.** At the start of the second turn, the Robber can be at any vertices in row and column 0 as well as vertices $v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,m-a-1}$. The Cop probes $B_2 = \{v_{m-1,m-1}, v_{m-2,m-2}, \ldots, v_{m-a,m-a}\}$. For this probe, there are four classes of distance vectors in $N[O_1]$:

$$\vec{D}(B_2, v_{x,y}) = \begin{cases} [2, 2, \ldots, 2] & \text{if } x = y = 0, \\ [3, 3, \ldots, 3] & \text{if } x = 0,\ 1 \le y < m-a \text{ and } 1 \le x < m-a,\ y = 0, \\ [3, \ldots, 3, 1, 3, \ldots, 3] & \text{if } x = 0,\ y = d \text{ and } x = d,\ y = 0, \\ [2, 4, 4, \ldots, 4] & \text{if } x = m-1, \end{cases}$$

where $d \in \{m-a, m-a+1, \ldots, m-1\}$. In the case where all the distances in the vector are three except one, the distance 1 is in the u'th position if $d = m - u$. From the distances it is clear that $v_{0,0}$ is resolved by B_2. Further, if $\vec{D}(B_2, v_{x,y}) = [2, 4, 4, \ldots, 4]$, then $O_2 = \{v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,m-a-1}\}$ such that the Cop wins by Lemma 6.2. If $\vec{D}(B_2, v_{x,y}) = [3, \ldots, 3, 1, 3, \ldots, 3]$ such that $O_2 = \{v_{d,0}, v_{0,d}\}$, the Cop wins by Lemma 6.4.

Lastly say $\vec{D}(B_2, v_{x,y}) = [3, 3, \ldots, 3]$. Now $O_2 = \{v_{0,e}, v_{e,0} \mid 1 \le e < m-a\}$ and $N[O_2]$ contains all vertices in rows and columns 1 to $m-a-1$ as well as $v_{0,0}$. Next, the Cop probes $B_3 = \{v_{1,1}, v_{2,2}, \ldots, v_{a,a}\}$. For this probe, there are the following classes of distance vectors:

$$\vec{D}(B_3, v_{x,y}) = \begin{cases} [3, \ldots, 3, 1, 3, \ldots, 3] & \text{if } x = 0,\ y = d \text{ and } x = d,\ y = 0, \\ [2, 2, 4, 4, \ldots, 4] & \text{if } x = f,\ y = g \text{ and } x = g,\ y = f, \\ [4, \ldots, 4, 2, 4, \ldots, 4] & \text{if } x > a,\ y = k \text{ and } x = k,\ y > a, \end{cases}$$

where $d \in \{1, 2, \ldots, m-a-1\}$, $f, g \in \{1, 2, \ldots, m-a-1\}$ and $k \in \{1, 2, \ldots, m-a-1\}$. In the first case, the d'th entry is equal to 1 and $O_3 = \{v_{d,0}, v_{0,d}\}$ such that the Cop wins by Lemma 6.4. Note that if $d = a+1$ when m is even, all distances are 3. In the second case, all entries are 4 except two, which is the f'th and g'th entries. It follows that the Robber is localized to $O_3 = \{v_{f,g}, v_{g,f}\}$ and the Cop wins by Lemma 6.5. In the third case, all entries are 4 except the e'th entry. Now safe sets of the form $O_3 = \{v_{a+1,k}, v_{a+2,k}, \ldots, v_{m-1,k}\} \cup \{v_{k,a+1}, v_{k,a+2}, \ldots, v_{k,m-1}\}$ where $k \in \{1, 2, \ldots, m-a-1\}$ and $N(O_3) = \{v_{a+1,0}, v_{a+2,0}, \ldots, v_{m-1,0}\} \cup \{v_{0,a+1}, v_{0,a+2}, \ldots, v_{0,m-1}\} \cup \{v_{k,0}, v_{0,k}\}$. Next the Cop probes $B_4 = \{v_{m-1,m-1}, v_{m-2,m-2}, \ldots, v_{m-a,m-a}\}$ such that three classes of distance vectors for safe sets exist:

$$\vec{D}(B_4, v_{x,y}) = \begin{cases} [3, \ldots, 3, 1, 3, \ldots, 3] & \text{if } x = 0,\ y = d \text{ and } x = d,\ y = 0;\ d > a, \\ [4, \ldots, 4, 2, 4, \ldots, 4] & \text{if } x = f,\ y = g \text{ and } x = g,\ y = f;\ f, g > a, \\ [3, 3, \ldots, 3] & \text{otherwise.} \end{cases}$$

In the first case, $O_4 = \{v_{d,0}, v_{0,d}\}$ and the Cop wins by Lemma 6.4. In the second, $O_4 = \{v_{f,g}, v_{g,f}\}$ and the Cop wins by Lemma 6.5. Lastly we have that $O_4 = \{v_{k,0}, v_{a+1,0}, v_{0,k}, v_{0,a+1}\}$. Note that if m is odd, this case is the same as the second case. In this case the Robber can be at any vertices in rows and columns k and $a+1$ as well as vertex $v_{0,0}$. Then the Cop probes B_5 such that it contains vertices $v_{m-1,k}$, $v_{m-1,a+1}$ and any $a-2$ other degree-2 vertices in column $m-1$. Since B_5 is similar to B_1, there are three classes of safe sets in $N[O_4]$. The first has distance vector $\vec{D}(B_5, r) = [4, \ldots, 4, 2, 4, \ldots, 4]$ and the Cop wins by Lemma 6.2. The second has distance vector $\vec{D}(B_5, r) = [3, 3, \ldots, 3]$ such that $O_5 = \{v_{k,0}, v_{a+1,0}\}$ and the Cop wins by Corollary 6.3. The third has distance vector $\vec{D}(B_5, r) = [4, 4, \ldots, 4]$ such that O_5 is similar to the case considered in Strategy 6.6.1.

The last case to consider after probe B_1, is $\vec{D}(B_1, v_{x,y}) = [3, 3, \ldots, 3]$:

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Strategy 6.6.3 $\vec{D}(B_1, v_{x,y}) = [3, 3, \ldots, 3]$. Here, $N[O_1]$ contains all vertices in columns 1 to $m-2$, all vertices in rows 1 to $m-a-1$ and vertex $v_{0,0}$. In the second turn, Cop probes $B_2 = \{v_{m-2,1}, v_{m-2,2}, \ldots, v_{m-2,a}\}$. Since B_2 is a translation of B_1, the possible distance vectors are the same as in Table 6.1. For the first three options, we already know that the Cop can guarantee a win. It follows that we only need to consider the case when $\vec{D}(B_2, r) = [3, 3, \ldots, 3]$. The robber set that corresponds to this distance vector is $O_2 = \{v_{1,0}, v_{2,0}, \ldots, v_{m-3,0}\} \cup \{v_{0,m-a-1}\}$, where vertex $v_{0,m-a-1}$ is excluded for odd m. Thus $N[O_2]$ contains all vertices in columns 1 to $m-3$ and all vertices in row $m-a-1$. The Cop probes $B_3 = \{v_{m-3,m-a-1}, v_{m-4,m-a-1}, \ldots, v_{m-a-2,m-a-1}\}$, which is a translation of B_1. Again the only distance vector to consider is $\vec{D}(B_3, r) = [3, 3, \ldots, 3]$. Now, $O_3 = \{v_{1,0}, v_{2,0}, \ldots, v_{m-a-3,0}\}$, where $m-a-3 < a$ and the Cop wins by Corollary 6.3.

This completes all cases of safe sets after probe B_1 and thus $\lceil \frac{m}{2} \rceil - 1$ cops are enough to guarantee a win for $m \geq 7$. □

6.2 Products of stars with small order

By Proposition 3.19 and Corollary 3.25, $2 \leq \zeta(S_m \Box S_m) \leq m-1$. Furthermore, if $m \in \{2, 3\}$, then $S_m = P_m$ such that $\zeta(S_m \Box S_m) = 2$. Thus we assume that $m \geq 4$. First we show that two cops are not enough to guarantee the Cop to win on the Cartesian product of two stars of order six.

Proposition 6.7. *For the star S_m of order $m = 6$, $\zeta(S_m \Box S_m) > 2$.*

Proof. Say the Cop probes two vertices on the star S_6. It is easy to see that for any choice of two probed vertices, at least three leaves of S_6 will belong to the same safe set in S_6. Therefore by Corollary 3.23, each row and column of $S_6 \Box S_6$ contain at least three vertices in the same safe set for any given two-vertex probe. Specifically, these three vertices will be in the same orbit. Therefore, after probe B_1, we assume without the loss of generality the existence of the following safe set: $O_1 = \{v_{1,0}, v_{2,0}, v_{3,0}\}$. Then, in the next turn, the Robber can be at any vertices in columns $1, 2$ and 3 as well as vertex $v_{0,0}$. If the Robber can continually stay in O_1, she can perpetually avoid capture. Thus, we assume that the Cop's probes resolves at least one of the vertices of O_1. This is only possible if at least one of the two vertices probed by the Cop is in columns 1 to 3. Therefore assume the Cop probes $B_2 = \{b_1, b_2\}$ such that $b_1 = v_{1,j}$ where $j \leq 5$. We consider two cases: $j = 0$ and $j \geq 1$. Since all leaves of S_6 are in the same orbit, we may assume in the second case that $j = 1$. The two cases as well as relevant distances are given in Figure 6.2.

We show that for all choices of b_2, one of three things will happen:

1. There will be a safe set in $N[O_1]$ containing vertex $v_{0,0}$.
2. There will be a safe set in $N[O_1]$ containing five vertices in the same row or column, excluding vertex $v_{0,0}$.
3. The neighbourhood of $N[O_2]$ will contain five vertices in column 0, excluding vertex $v_{0,0}$.

Case 6.7.1 $b_1 = v_{1,0}$. Say $b_2 = v_{k,l}$. If $l = 0$, then $\vec{D}(B_2, v_{1,y}) = [1, \alpha]$ for some fixed integer α and $y \geq 1$. Thus a safe sets exists consisting of five vertices in the same column. Now assume $l \geq 1$. If $k = 0$, we have that $\vec{D}(B_2, v_{0,0}) = [1, 1] = \vec{D}(B_2, v_{1,l})$ such that $v_{0,0}$ is a safe vertex. If $k \geq 2$, we have that $\vec{D}(B_2, v_{0,0}) = [1, 2] = \vec{D}(B_2, v_{1,l})$ and again $v_{0,0}$ is in a safe set. Finally for $k = 1$, $\vec{D}(B_2, v_{0,0}) = [1, 2]$ and $\vec{D}(B_2, v_{1,q}) = [1, 2]$ for $q \neq l, 0$. Hence $v_{0,0}$ is in a safe set.

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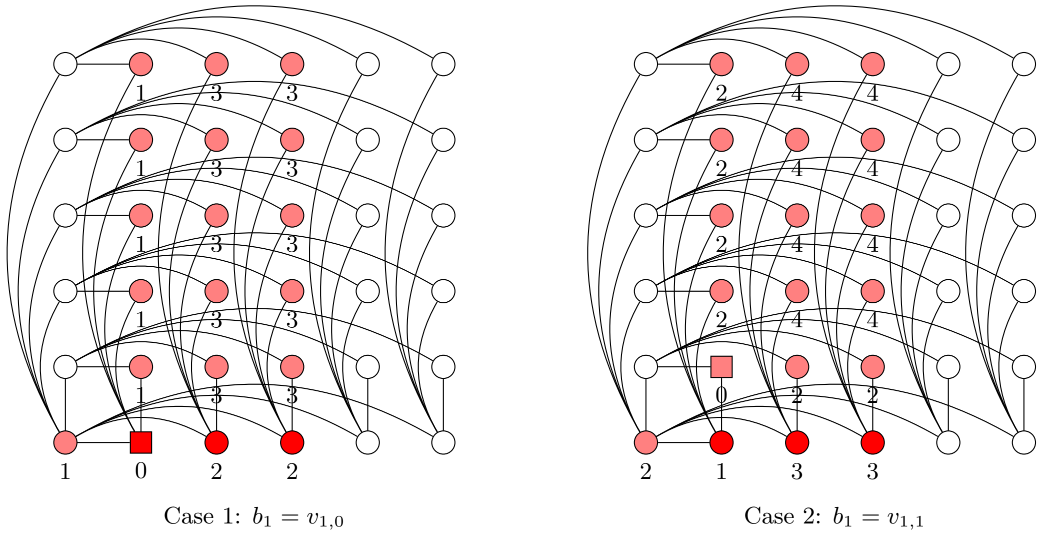

FIGURE 6.2: *The case two cases for b_1 in $B_2 = \{b_1, b_2\}$ in Proposition 6.7.*

Case 6.7.2 $b_1 = v_{1,1}$. First say $b_2 = v_{0,0}$. Now, $\vec{D}(B_2, v_{1,b}) = [2,2]$ for $b \geq 2$ and $\vec{D}(B_2, v_{a,1}) = [2,2]$ for $a = 2, 3$. Let $O_2 = \{v_{a,1}, v_{1,b} \mid a = 2,3,\ b > 1\}$, then the neighbourhood $N[O_2]$ contains five vertices in column 0. Now, let $B_2 = \{v_{1,1}, v_{k,l}\}$ where $k = l = 0$ does not occur. If $k = 0$ and $l \neq 0$, then $\vec{D}(B_2, v_{0,0}) = [2,1]$. If $l \geq 2$, we have $\vec{D}(B_2, v_{1,l}) = [2,1]$. If $l = 1$, we have $\vec{D}(B_2, v_{r,1}) = [2,1]$ for $r = 2, 3$. In both cases, $v_{0,0}$ is a safe vertex. Now assume $k \geq 1$. If $k = 1$ and $l = 0$, we have that $\vec{D}(B_2, v_{0,0}) = [2,1] = \vec{D}(B_2, v_{1,b})$ for $b \geq 2$ such that $v_{0,0}$ is a safe vertex. If $k = 1$ and $l \geq 2$, we have that $\vec{D}(B_2, v_{0,0}) = [2,2]$, where this distance is shared by at least one other vertex in column 1 and thus $v_{0,0}$ is a safe vertex. Thus, assume $k \in \{2,3\}$. If $l = 0$, then $\vec{D}(B_2, v_{0,0}) = [2,1] = \vec{D}(B_2, v_{k,1})$. If $l = 1$, then $\vec{D}(B_2, v_{0,0}) = [2,2] = \vec{D}(B_2, v_{q,1})$ where $q \in \{2,3\}$ and $q \neq k$. If $l \geq 2$, then $\vec{D}(B_2, v_{0,0}) = [2,2] = \vec{D}(B_2, v_{k,1})$. In each case $v_{0,0}$ is a safe vertex. Lastly, say $k \geq 4$. If $l = 0$, then $\vec{D}(B_2, v_{1,z}) = [2,3] = \vec{D}(B_2, v_{p,1})$ where $z \geq 2$ and $p \in \{2,3\}$. If $O_2 = \{v_{1,z}, v_{p,1} \mid z \geq 2, p \in \{2,3\}\}$, then $N[O_2]$ contains at least five vertices in column 0. If $l = 1$, then $\vec{D}(B_2, v_{0,0}) = [2,2] = \vec{D}(B_2, v_{d,1})$ where $d \in \{2,3\}$. If $l \geq 2$, then $\vec{D}(B_2, v_{0,0}) = [2,2] = \vec{D}(B_2, v_{1,l})$. In both cases $v_{0,0}$ is a safe vertex.

For these three outcomes the Robber can always use one of the following strategies to return to a safe set containing three vertices in row 0, as in O_1:

Strategy 6.7.1 Safe set O contains at least five vertices in the same row or column. If O does not contain $v_{0,0}$, then it follows from Corollary 3.23 that the $N[O]$ contains a safe set in row or column 0 with at least three vertices, as in O_1.

Strategy 6.7.2 Safe set O contains $v_{0,0}$. In this case, $N[O]$ will contain all vertices in column and row 0. Thus, irrespective of the Cop's next probe, at least three vertices in column and row 0 will form part of the same safe set as in O_1.

Strategy 6.7.3 Safe set O such that $N[O]$ contains at least five vertices in column 0. The Robber follows Strategy 6.7.1 in the third turn.

Hence the Robber can always avoid detection if two cops are used on $S_6 \Box S_6$. $\square$

In the following result it is shown that $\left\lfloor \frac{m}{2} \right\rfloor$ cops are enough for $m = 4, 5$ and 6.

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Proposition 6.8. *For the star S_m or order $m \in \{4,5,6\}$, $\zeta(S_m \square S_m) \leq \lfloor \frac{m}{2} \rfloor$.*

Proof. Let $m = 6$. In the first turn, the Cop probes $B_1 = \{v_{5,5}, v_{5,4}, v_{5,3}\}$ such that there are four classes of safe sets as in Table 6.1.

Strategy 6.8.1 $\vec{D}(B_1, r) = [2,4,4]$**.** The cases when $\vec{D}(B_1, r) = [4,2,4]$ and $[4,4,2]$ follows similarly. For the robber set $O_1 = \{v_{1,5}, v_{2,5}, v_{3,5}, v_{4,5}\}$, $N(O_1) = \{v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}\} \cup \{v_{0,5}\}$. The Cop probes $B_2 = \{v_{2,5}, v_{3,5}, v_{4,5}\}$ and wins since B_2 is a translation of B_1.

Strategy 6.8.2 $\vec{D}(B_1, r) = [4,4,4]$**.** For the robber set $O_1 = \{v_{i,1}, v_{i,2} \mid i = 1,2,3,4\}$ we have $N(O_1) = \{v_{0,1}, v_{0,2}, v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}\}$. The Cop probes $B_2 = \{v_{2,2}, v_{3,2}, v_{4,2}\}$, which is a translation of B_1. It follows that the Robber is localized to $O_2 = \{v_{1,0}, v_{0,1}\}$ and can be at any vertices in row and column 1 as well as vertex $v_{0,0}$. The Cop probes $B_3 = \{v_{5,1}, v_{5,2}, v_{5,3}\}$, again a translation of B_1. There are two safe sets in $N[O_2]$: $\{v_{1,5}, v_{1,4}\}$ and $\{v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}\}$. Both can be solved by Strategy 6.8.1 and thus the Cop wins.

Strategy 6.8.3 $\vec{D}(B_1, r) = [2,2,2]$**.** The Robber is localized to $O_1 = \{v_{0,0}, v_{5,1}, v_{5,2}\}$ such that $N[O_1]$ contains all vertices in row and column 0 as well as $v_{5,1}$ and $v_{5,2}$. The Cop probes $B_2 = \{v_{5,5}, v_{4,4}, v_{3,3}\}$, localizing the Robber to one of three cases of safe sets:

1. $O_2 = \{v_{5,1}, v_{5,2}\}$ and the Cop wins by Strategy 6.8.1.
2. $O_2 = \{v_{k,0}, v_{0,k}\}$, $k \in \{3,4,5\}$ and the Cop probes B_3 in Strategy 6.8.2.
3. $O_2 = \{v_{1,0}, v_{2,0}, v_{0,1}, v_{0,2}\}$.

In the last case, $N[O_2]$ contains all vertices in rows and columns $1,2$ as well as vertex $v_{0,0}$. The Cop now probes $B_3 = \{v_{5,1}, v_{5,2}, v_{5,3}\}$ such that their are four types of safe sets:

1. $O_3 = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}\}$, $j \in \{1,2\}$ and the Cop wins with Strategy 6.8.1.
2. $O_3 = \{v_{1,3}, v_{2,3}\}$ and the Cop wins with Strategy 6.8.1.
3. $O_3 = \{v_{1,4}, v_{2,4}, v_{1,5}, v_{2,5}\}$ and the Cop wins with Strategy 6.8.2.
4. $O_3 = \{v_{1,0}, v_{2,0}\}$.

In the last case, $N[O_3]$ contains all vertices in columns $1,2$ as well as vertex $v_{0,0}$. If the Cop probes $B_4 = \{v_{1,5}, v_{2,5}, v_{3,5}\}$, he wins by Strategy 6.8.1 in the next turn.

Strategy 6.8.4 $\vec{D}(B_1, r) = [3,3,3]$**.** The Robber is now localized to robber set $O_1 = \{v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}, v_{0,1}, v_{0,2}\}$ and thus $N[O_1]$ contains all vertices in rows $1,2$, columns 1 to 4 and vertex $v_{0,0}$. The Cop probes $B_2 = \{v_{4,1}, v_{4,2}, v_{4,3}\}$ such that there are four types of safe sets as in the first turn. The first three types were solved in the previous three strategies. The fourth type, given by $\vec{D}(B_2, r) = [3,3,3]$, defines robber set $O_2 = \{v_{1,0}, v_{2,0}, v_{3,0}\}$. Here the Cop wins in the next turn by probing $B_3 = \{v_{1,5}, v_{2,5}, v_{3,5}\}$ and thereafter applying Strategy 6.8.1.

Results follow in a similar fashion for $m = 4$ and 5. $\square$

By Theorem 6.10, Equation (6.3) does not hold for $S_6 \square S_6$, because $\zeta(C_6) = 2 > \zeta(S_6) = 1$ where $\zeta(C_6 \square C_6) = 2 < \zeta(S_6 \square S_6)$. Thus possibly contrary to intuition, the following proposition follows:

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Proposition 6.9. *There exist graphs G and H such that $\zeta(G) < \zeta(H)$ and $\zeta(G\Box G) > \zeta(H\Box H)$.*

We have now proven the following result:

Theorem 6.10. *Let S_m be a star of order $m \geq 4$. Then $\zeta(S_m\Box S_m) = \left\lfloor \frac{m}{2} \right\rfloor$ for $m \in \{4, 5, 6\}$ and $\zeta(S_m\Box S_m) = \left\lceil \frac{m}{2} \right\rceil - 1$ for $m \geq 7$.*

Note that since $\zeta(S_n) = 1$ as by Equation (2.3), it follows from Corollary 3.25 that

$$\zeta(G_m\Box S_n) \leq \psi(G_m) \leq m - 1. \tag{6.5}$$

6.3 Chapter summary

In this chapter, we obtained the localization number of the product of two stars of the same order. In doing so, we showed that the difference between $\zeta(G\Box G)$ and $\zeta(G)$ can be arbitrarily large. We also showed that $\zeta(G) < \zeta(H)$ does not always imply $\zeta(G\Box G) \leq \zeta(H\Box H)$.

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CHAPTER 7

Conclusion

A summary of work done in this thesis is given in Section 7.1 and possible future work is discussed in Section 7.2.

7.1 Thesis summary

The first chapter gave an introduction to the localization game, as well as some basic definitions and an overview of this thesis. Then, in Chapter 2, a review of current results in literature was given. It started off with a summary of the game of Cops and Robbers, as this is the predecessor of the localization game. Then the robber locating game as well as the backtrack robber locating game was considered. The former is a variant of the localization game and the latter is the localization game played with only one cop. After this the known results on the localization game were discussed and the chapter concluded with a review of another variant of the localization game, namely the centroidal localization game.

In Chapter 3 an example game and basic results were given as a warm-up. The localization number was determined for complete graphs, cycles and grids. In Section 3.3 bounds on the localization number of Cartesian products were provided. This included proving that the localization number of the product of two graphs can never be less than the localization number of the individual graphs. It was also shown that one less cop than the sum of the localization number of the one graph and the doubly resolving number of the other will always be enough to guarantee a win for the Cop on the product of two graphs. The chapter ended with two results on the doubly resolving number of a graph. The work done in this chapter as well the result on the localization number of the product of cycles has been submitted for publication [6].

Products with complete graphs were investigated in Chapter 4. In Section 4.1 it was shown that m cops are always enough for the Cop to win on the product of any graph with a complete graph, both of orders m. Further if the two graphs do not have the same order, then one less cop than the maximum order is enough to win. In the next two sections, the localization number of the product of two complete graphs and the product of a complete graph with a cycle was determined.

Chapter 5 focused on the product of cycles. It was shown that the localization number of the product of two cycles is mostly equal to two. Three cases were considered: odd by odd, odd by even and even by even. The last section of this chapter contained results on the localization number of the product of a path and a cycle as well as an upper bound to the localization number of the product of an arbitrary graph with a cycle.

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Lastly the localization number of the product of two stars of the same order were determined. This consisted of two cases: stars of order at least seven and stars of order at most six. In doing so it was shown that the difference between the localization number of the product of two copies of the same graph and the localization number of the single graph can be arbitrarily large. Further it was also illustrated that there exists two graphs G and H such that the localization number of G is less than that of H, but the localization number of $G\Box G$ is larger than that of $H\Box H$.

7.2 Future work

Let each of G and H be either a path, a cycle, a compete graph or a star of orders m and n respectively, with $m \geq n$. It was shown in this thesis that $\zeta(G\Box H) \leq m$ and that $\zeta(G\Box H) = m$ if $G = H = K_m$. Since $\zeta(G) \leq \zeta(K_m)$ and $\zeta(K_m\Box K_m) = m$, it is natural to ask the following question:

Question 7.1. *Let G and H be any connect graphs of order m. Is it true that $\zeta(G\Box H) \leq m$?*

Further since $\zeta(G) \leq m-1$ for any connected graph G and $\zeta(G) = m-1$ if and only if $G = K_m$, it would be worthwhile to investigate if a similar result is true for Cartesian products.

Question 7.2. *Let G and H be any connect graphs of order m. Is it then true that $\zeta(G\Box H) = m$ if and only if $G = H = K_m$?*

Question 7.1 is true if G is a path, cycle, complete graph or a star as shown in this thesis. In general from Theorem 3.24 we know that

$$\zeta(G\Box H) \leq 2m - 3 \tag{7.1}$$

since $\zeta(G) \leq m-1$ and $\psi(H) \leq m-1$. It therefore follows that Question 7.1 is true for $m = 3$. Question 7.1 is also true if $G = K_m$. Therefore if G is not complete, $\zeta(G) \leq m-2$. It then follows that Equation (7.1) can be improved to $\zeta(G\Box H) \leq 2m-4$; thus proving Question 7.1 holds for $m = 4$. By Corollary 3.6, $\zeta(G\Box H) \leq 3$ if $\Delta \leq 3$ and thus Question 7.1 holds for $m \geq 5$ and $\Delta \leq 3$. It follows that $G\Box H$ only needs to be considered when $m \geq 5$ and $\Delta \geq 4$. In fact, the question will hold true as long as $\zeta(G\Box H) \leq \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor + 1 \leq m$ by Proposition 2.1.

Note that if the diameters of G and H are at least three, then $\zeta(G\Box H) \leq 2m-5$ by Proposition 3.29 such that Question 7.1 holds for $m = 5$.

The following proposition gives us information about safe sets in $G\Box H$ for any connected G and H with m cops:

Proposition 7.3. *Let G and H be any connected graphs of order $m \geq 5$. Then the Cop can play with m cops on $G\Box H$ such that $|O_t| = 2$, where O_t is the robber set in turn $t \geq 2$.*

Proof. In the first turn, the Cop probes B_1 such that it contains m rows and columns. Then by Corollary 3.23, the robber set O_1 can contain at most m vertices, where each vertex is in a unique row and column. In the next turn, the Cop probes B_2 such that $O_1 \subseteq B_2$ and at least $m-1$ unique rows and columns are probed. This is possible, since $|O_1| \leq m$ and further since $m-1$ rows and columns are probed, Corollary 3.23 can be applied again. Say vertices $x, y \in N[O_1]$ belong to the same safe set after probe B_2. Since $x \in N[O_1]$ and $O_1 \subseteq B_2$, it follows that $d(b_2, x) = 1$ for some $b_2 \in B_2$. Thus x is either in the same row or column as b_2, say

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the same column. If $d(b_2, y) = d(b_2, x)$, then y is in the same row as b_2. Further, O_2 can only contain two vertices by Corollary 3.23. In following turns, the Cop repeats this strategy such that $|O_t| = 2$, where both vertices in O_t are adjacent to one vertex in O_{t-1}. □

Note that this does not prove that only two safe vertices exists after probe B_t, but rather that all safe sets after a probe are safe pairs. Considering the proposition, the following question can be asked:

Question 7.4. *Let G and H be any connect graphs of order m. What is the smallest non-negative integer c such that $\zeta(G \square H) \leq m + c$?*

In Theorem 4.4 it was shown that the upper bound to $\zeta(G_m \square K_m)$ was smaller than that of $\zeta(G_m \square K_n)$ if $m \neq n$, leading to the following question:

Question 7.5. *Let G and H be connect graphs of orders m and n respectively, where $m \neq n$. Is it true that $\zeta(G \square H) \leq \max\{m, n\} - 1$?*

In Chapter 3 it was shown that every Cartesian product contains a hideout if one cop is used. For the backtrack robber locating game there exist classes of subgraphs that are hideouts if one cop is used.

Question 7.6. *Does there exist certain subgraphs that are hideouts if two cops are used? In other words, does there exist graph classes such that if G contains such a graph as a subgraph, then $\zeta(G) > 2$?*

Considering the doubly resolving number of a graph, in Section 3.4 it was shown that $\psi(S_m) = m - 1$ and that $\psi(G_m) \leq m - 2$ if G has a diameter of at least three.

Question 7.7. *For which classes of graphs other than complete graphs and stars is it true that $\psi(G_m) = m - 1$?*

In Chapter 5 it was shown that $\zeta(G_m \square C_n) \leq m - 1$ for all cases but $n \in \{4, 6\}$, in which case the upper bound is m. Can we improve on this bound?

Question 7.8. *Let G_m be a connected graph of order $m \geq 4$ and C_n be a cycle of order $n \in \{4, 6\}$. Is it true that $\zeta(G_m \square C_n) \leq m - 1$?*

In Chapter 6 only stars of the same order were considered. It will be interesting to see how the localization number changes if the Cartesian product of stars and other graphs are investigated.

Question 7.9. *Let S_n be the star of order n. What is $\zeta(G \square S_n)$, where $G \in \{S_m, P_m, C_m, K_m\}$?*

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